Homework 8, Due Wednesday, October 30, 2002

Problems on Sections 3.1-3.6
1. HV Problem 3.23 (b), (c), (e).
2. HV Problem 3.24 (c), (e).

Problems on Sections 4.1-4.4

Note: solving some of these problems (e.g., 4.2, 4.3(b), 4.3(c)) may require use of partial fraction expansions. These are covered at the level required here in Examples 3.20 and 3.21 on p. 200 and 202, respectively. More complete discussion is given in Appendices B.1 and B.2, which we will cover when we study Laplace and Z transforms.

3. HV Problem 4.1 (b), (d).
4. HV Problem 4.2 (a).
5. HV Problem 4.3 (b), (c).
6. HV Problem 4.12. In part (b) (iii), choose RC so that the peak value of the component at 240\(\pi\) rad/s is less than 1% of the average value.

Note: we could also solve this particular problem using FS, but you are asked to use FT here.

Note: if the full-wave rectifier were a linear, time-invariant system, we would be able to say that

\[ G(j\omega) = F(j\omega)X(j\omega) \]

where \( F(j\omega) \) represented the frequency response of the system. However, \( x(t) \) contains power only at \( \omega = \pm 120\pi \), whereas \( g(t) \) contains power at many harmonics of those frequencies. In other words, the full-wave rectifier is a nonlinear, time-invariant system, and

\[ G(j\omega) \neq 0 \]

at some values of \( \omega \) at which \( X(j\omega) = 0 \), so that \( G(j\omega) \neq F(j\omega)X(j\omega) \).

7. HV Problem 4.13 (c).
8. HV Problem 4.15 (c).
9. HV Problem 4.16 (b).
10. This problem explains how the group delay of a LTI system affects baseband signals (signals whose FT is nonzero only near d.c.). Consider a continuous-time, LTI system having a real impulse response \( h(t) \) and a frequency response \( H(j\omega) \). We write the frequency response in polar form as

\[ H(j\omega) = |H(j\omega)|e^{j\arg\{H(j\omega)\}} = |H(j\omega)|e^{j\theta(\omega)} \]

The input signal is \( x(t) \leftrightarrow X(j\omega) \). Assume that \( X(j\omega) \) is nonzero only near \( \omega = 0 \). Over the range of \( \omega \) where \( X(j\omega) \) is nonzero, \( H(j\omega) \) varies slowly, and we can expand \( H(j\omega) \) in a Taylor series about the point \( \omega = 0 \):
Show that, including terms up to first order in $\omega$, the output is given approximately by:

$$
y(t) = \left| H(j0) \right| x(t - \tau_g(0)) ,
$$

where $\tau_g(0) = -\left. \frac{d\phi(\omega)}{d\omega} \right|_{\omega = 0}$. Thus, $y(t)$ is approximately a scaled, delayed version of $x(t)$.

**Hint:** start by representing $y(t)$ as an inverse Fourier transform of $H(j\omega)X(j\omega)$.

### MATLAB Exercises on Sections 4.1-4.4

Work the portion of HV Section 4.12 on the Frequency Response of LTI Systems. You need not turn in any work on Section 4.12. Turn in your work on the following two problems.

11. This problem explores amplitude and phase distortion in a continuous-time LTI system. See the handout on “Examples of LTI Filtering of Periodic Signals”, which was given out on September 30. Choose the periodic input signal $x(t)$ to be the same square wave as in that handout, with $T = 1$, $T_s = 1/8$. Consider a first-order lowpass filter $h(t) = (RC)^{-1}e^{-t/RC}u(t)$ $\leftrightarrow$ $H(j\omega) = 1/(1 + j\omega RC)$. For each case below, plot a partial sum of the FS representation of one period of the signal of interest, including terms from $k = -128$ to 128. As an example, for the input signal $x(t)$, this partial sum is:

$$
\hat{x}_{128}(t) = \sum_{k = -128}^{128} X[k]e^{jk\omega_0 t}.
$$

For each case, plot one period of the signal using a time resolution of 0.001, i.e.,

$t = -0.5:0.001:0.5$.

- a. Plot $\hat{x}_{128}(t)$, the partial sum of the FS representation of the input.

- b. Plot $\hat{y}_{128}(t)$, the partial sum of the FS representation of the output, when the LTI system has frequency response $H(j\omega) = 1/(1 + j\omega RC)$, i.e., it includes both the amplitude and phase distortion of the lowpass filter. Consider $RC = 0.03$ and 0.3. In each case, evaluate by eye the delay of the centroid of the pulse, and compare it to the low-frequency group delay $\tau_g(0) = RC$. According to Problem 10, these should be approximately equal.

- c. Plot $\hat{y}_{128}(t)$, the partial sum of the FS representation of the output, when the LTI system has frequency response $H_{\text{amp. only}}(j\omega) = 1/\sqrt{1 + (\omega RC)^2}$, i.e., it includes only the amplitude distortion of the lowpass filter. Consider $RC = 0.03$ and 0.3.

- d. Plot $\hat{y}_{128}(t)$, the partial sum of the FS representation of the output, when the LTI system has frequency response $H_{\text{phs. only}}(j\omega) = e^{-\tan^{-1}(\omega RC)}$, i.e., it includes only the phase distortion of the lowpass filter. Consider $RC = 0.03$ and 0.3.
12. This problem explores the use of “window functions” to reduce the Gibbs phenomenon that occurs in summing Fourier series of signals having discontinuities. Consider a periodic signal $x(t)$, which we can represent using the FS:

$$x(t) \leftrightarrow_{\omega_0} X[k].$$

The FS synthesis equation tells us how to synthesize $x(t)$:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}.$$ 

We know that if the original waveform $x(t)$ has discontinuities at some values of $t$, the synthesis $\hat{x}(t)$ will exhibit ringing and overshoot near those values of $t$. Consider a LTI system described by:

$$w(t) \leftrightarrow W(j\omega).$$

We assume that the impulse response $w(t)$ is aperiodic. In this context, we refer to the frequency response $W(j\omega)$ as a “window function”. Consider the following diagram, which describes what happens when the periodic signal $x(t)$ is input to the system.

$$x(t) \leftrightarrow_{\omega_0} X[k] \xrightarrow{w(t) \leftrightarrow W(j\omega)} w_w(t) = w(t)^{*}x(t) \leftrightarrow_{\omega_0} W(jk\omega_0)X[k]$$

From our study of the filtering of periodic signals by LTI systems, we know that the output $x_w(t) = w(t)^{*}x(t)$ has FS coefficients $W(jk\omega_0)X[k]$. Thus, we can synthesize $x_w(t)$ using:

$$\hat{x}_w(t) = \sum_{k=-\infty}^{\infty} W(jk\omega_0)X[k]e^{jk\omega_0 t}.$$ 

If we choose the window function properly, the synthesized waveform $\hat{x}_w(t)$ can exhibit less ringing and overshoot than $\hat{x}(t)$. Here we will consider two choices of the window function.

**Rectangular Window**

Here the window function is:

$$W_{rect}(j\omega) = \begin{cases} 1 & |\omega| \leq J\omega_0 \\ 0 & |\omega| > J\omega_0 \end{cases},$$

$$W_{rect}(jk\omega_0) = \begin{cases} 1 & |k| \leq J \\ 0 & |k| > J \end{cases}.$$ 

When we synthesize the waveform using the rectangular window function, we simply sum the FS for $-J \leq k \leq J$. The synthesized signal $\hat{x}_w(t)$ is simply $x(t)$ convolved with the inverse FT of $W_{rect}(j\omega)$, which is:
The triangular window function is given by:

\[ w_{\text{rect}}(t) = \frac{J\omega_0}{\pi} \text{sinc}\left(\frac{J\omega_0}{\pi} t\right). \]

**Triangular Window**

The triangular window function is given by:

\[
W_{\text{tri}}(j\omega) = \begin{cases} 
1 - \frac{|\omega|}{J\omega_0} & |\omega| \leq J\omega_0 \\
0 & |\omega| > J\omega_0 
\end{cases}.
\]

\[
W_{\text{tri}}(jk\omega_0) = \begin{cases} 
1 - \frac{|k|}{J} & |k| \leq J \\
0 & |k| > J 
\end{cases}.
\]

When we synthesize using the triangular window function, we sum the FS for \(-J \leq k \leq J\), but scale each FS coefficient by a linear function that goes to zero when \(k = \pm J\). In this case, \(\hat{x}_w(t)\) is \(x(t)\) convolved with:

\[ w_{\text{tri}}(t) = \frac{J\omega_0}{2\pi} \text{sinc}^2\left(\frac{J\omega_0}{2\pi} t\right). \]
The rectangular and triangular window functions and their inverse FTs are compared in the figure above. Note that the central lobe of \( w_{\text{rect}}(t) \) is narrower and taller than that of \( w_{\text{tri}}(t) \), so using the rectangular window will result in \( \hat{x}_w(t) \) having sharper rising and falling edges. However, \( w_{\text{rect}}(t) \) has larger sidelobes than \( w_{\text{tri}}(t) \), so using the rectangular window will result in \( \hat{x}_w(t) \) having more ringing and overshoot at values of \( t \) where \( x(t) \) is discontinuous.

**MATLAB Exercise**

See the handout on “Examples of LTI Filtering of Periodic Signals”, which was given out on September 30. Choose \( x(t) \) to be the same square wave as in that handout, with \( T = 1 \), \( T_s = 1/8 \). For both the rectangular and triangular windows, for \( J = 7, 29, 99 \), plot \( \hat{x}_w(t) = \sum_{k=-\infty}^{\infty} W(jk\omega_0)X[k]e^{jk\omega_0 t} \) over a single period using a time resolution of 0.001, i.e., \( t = -0.5:0.001:0.5 \). For each of these six cases, also plot the \( w(t) \) on the same time scale. In each case try to visualize \( \hat{x}_w(t) \) as \( w(t)^*x(t) \).