Midterm 1

- The exam is for one hour and 50 minutes.
- The maximum score is 100 points. The maximum score for each part of each problem is indicated.
- The exam is closed-book and closed-notes. Calculators, computing, and communication devices are NOT permitted.
- Two double-sided sheets of notes are allowed. These should be legible to normal eyesight, i.e. the lettering should not be excessively small.
- No form of collaboration between students is allowed.

1. (10 points) State whether the following are true or false. In each case, give a brief explanation. A correct answer without a correct explanation gets 2 points. A correct answer with a correct explanation gets 5 points.

   (a) Let \( x(t) \) be a continuous time signal. Let \( y_1(t) \), \( y_2(t) \), and \( y_3(t) \) denote the respective outputs of a causal linear time invariant system to the inputs \( x(t) \), \( x^2(t) \), and \( x^3(t) \). Then \( y_3(t) \) can be determined from \( y_1(t) \) and \( y_2(t) \).

   (b) If \( x[n] \) is a nonnegative sequence with discrete time Fourier transform \( X(e^{j\omega}) \), then

\[
\sum_{n=-\infty}^{\infty} x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})| \, d\omega.
\]

2. (10 points)

Let \( x[n] \) and \( y[n] \) be periodic with period 3, with

\[
x[n] = \begin{cases} 
1 & \text{if } n = -1 \\
2 & \text{if } n = 0 \\
1 & \text{if } n = 1 
\end{cases}
\]

and

\[
y[n] = \begin{cases} 
-1 & \text{if } n = -1 \\
2 & \text{if } n = 0 \\
1 & \text{if } n = 1 
\end{cases}
\]

Let \( z[n] \) be the periodic sequence of period 3 that is the periodic convolution of \( x[n] \) and \( y[n] \), i.e.

\[
z[n] = \sum_{l\in\mathbb{Z}} x[l]y[n-l].
\]

Determine \( z[n] \).
3. (10 points)

Let

\[ \Lambda(t) = \begin{cases} 
1 - |t| & \text{if } |t| \leq 1 \\
0 & \text{otherwise} 
\end{cases} \]

Let

\[ x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t - 2n) \cos(\omega_0 t) . \]

Find the Fourier transform of \( x(t) \).

**Hint:** The function

\[ z(t) = \begin{cases} 
1 & \text{if } |t| \leq \frac{1}{2} \\
0 & \text{otherwise} 
\end{cases} \]

has Fourier transform

\[ Z(j\omega) = \text{sinc}(\frac{\omega}{2\pi}) . \]

4. (10 points) Let \( x[n] \) be a periodic sequence with period \( N \). Assume \( N = 3K \) for some integer \( K \). Let \( a_k \) denote the discrete time Fourier series coefficients of \( x[n] \). If \( a_k = 0 \) when \( k \) is not a multiple of 3, show that \( x[n] \) must also be periodic with period \( K \).

5. (5 + 5 points) Consider the causal linear time-invariant system whose input and output are related by the difference equation

\[ y[n] + \frac{1}{3} y[n - 1] = x[n] + x[n - 2] - 3x[n - 5] . \]

(a) Find the transfer function of the system.

(b) Find the output of this system for the input

\[ x[n] = (-1)^n \text{ for all } n \text{ .} \]

6. (5 + 5 + 5 points) Consider the continuous time system whose output \( y(t) \) for the input \( x(t) \) is given by

\[ y(t) = x(t) - \left( \int_{t}^{t+1} x(u)du \right)^2 . \]

Is it :

(a) linear ?

(b) causal ?

(c) BIBO stable ?

In each case explain your answers briefly. There should be no ambiguity about which part of the problem you are answering. A correct answer with an incorrect explanation will get at most 2 points.
7. (7 + 8 points) Let
\[ x_1(t) = \begin{cases} 
1 & \text{if } |t| \leq \frac{1}{2} \\
0 & \text{otherwise} 
\end{cases} 
\]
and
\[ x_2(t) = \begin{cases} 
t & \text{if } 0 \leq t \leq 1 \\
0 & \text{otherwise} 
\end{cases} 
\]
(a) Sketch \( y(t) = x_1(t) \ast x_2(t) \). You DO NOT need to write any formulas. The shape of your sketch of \( y(t) \) should be accurate and the coordinates should be properly marked.
(b) Let
\[ z_1(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 1 \\
3 & \text{if } 1 < t \leq 2 \\
0 & \text{otherwise} 
\end{cases} 
\]
and let
\[ z_2(t) = \begin{cases} 
t & \text{if } 0 \leq t \leq 1 \\
\frac{1}{3} t - \frac{1}{3} & \text{if } 1 < t \leq 2 \\
0 & \text{otherwise} 
\end{cases} 
\]
Let \( w(t) = z_1(t) \ast z_2(t) \). Determine \( w(t) \) in terms of \( y(t) \) using basic properties of convolution. You need not determine \( w(t) \) explicitly: just write it in terms of \( y(t) \).

8. (10 + 10 points) Consider the function
\[ x(t) = \begin{cases} 
1 & \text{if } -1 < t \leq 0 \\
1 + t & \text{if } 0 < t \leq 1 \\
2 - t & \text{if } 1 < t \leq 2 \\
0 & \text{otherwise} 
\end{cases} 
\]
(a) Find the Fourier transform \( X(j\omega) \) of the function \( x(t) \).
(b) Let \( y(t) \) be the periodic function of period 8 defined by
\[ y(t) = \begin{cases} 
x(t - \frac{3}{2}) & \text{if } 0 < t \leq 4 \\
-x(t + \frac{1}{2}) & \text{if } -4 < t \leq 0 
\end{cases} 
\]
Find the Fourier series coefficients of \( y(t) \).