Sample Final

- The exam is for three hours.

- The maximum score is 100 points. The maximum score for each part of each problem is indicated.

- The exam is closed-book and closed-notes. Calculators, computing, and communication devices are NOT permitted.

- Six double-sided sheets of notes are allowed. These should be legible to normal eyesight, i.e. the lettering should not be excessively small.

- No form of collaboration between students is allowed.

1. *(4 points)* State whether the following is true or false. Give a brief explanation. A correct answer with a correct explanation gets 4 points. A correct answer without a correct explanation gets only 1 point.

   - A causal linear time invariant continuous time system with a rational Laplace transform which has more zeros than poles in the right half plane is stable.

2. *(5 points)*

   Let \( x(t) \geq 0 \) and \( y(t) \geq 0 \) be nonnegative continuous time signals, with respective Fourier transforms \( X(j\omega) \) and \( Y(j\omega) \). A Stanford student claims that

   \[
   \int_{-\infty}^{\infty} |X(j\omega) + Y(j\omega)|^2 \, d\omega \geq \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega .
   \]

   Is this true or false? Either explain clearly why this is true or give a counterexample.

3. *(5 + 10 points)* A real valued continuous time signal \( x(t) \) is known to be bandlimited: if \( X(j\omega) \) denotes the Fourier transform of \( x(t) \), then \( X(j\omega) = 0 \) for \( |\omega| \geq W \).

   The signal is nonuniformly sampled with the pulse train

   \[
   q(t) = \sum_{n=-\infty}^{\infty} \left[ \delta(t - nT) + \delta(t - nT - \frac{T}{4}) \right].
   \]

   To begin with, we do not assume that there is any special relation between \( W \) and \( T \).

   (a) Let \( y(t) = x(t)q(t) \). Determine the Fourier transform of \( y(t) \), i.e. \( Y(j\omega) \).

   (b) Find the best possible condition you can on the relation between \( W \) and \( T \) which will ensure that it is possible to perfectly recover \( x(t) \) from \( y(t) \).
4. (12 points)
Consider the linear constant coefficient differential equation
\[
\frac{d^2}{dt^2} y(t) - y(t) = \frac{d}{dt} x(t),
\]
thought of as giving the output function \( y(t) \) in terms of the input function \( x(t) \).
Find a linear time invariant anticausal system whose input and output are related by this differential equation.

5. (10 points)
A discrete time linear time invariant system is such that the input
\[
x[n] = \delta[n] - \delta[n - 1] + \frac{1}{4} \delta[n - 2],
\]
produces the output
\[
y[n] = \frac{1}{2^{n-2}} u[n - 2].
\]
Is the system stable? Carefully and clearly explain your answer.

6. (12 points)
Let \( x[n] \) be a discrete time signal. We first downsample it by a factor of 2, resulting in the signal
\[
x_1[n] = x[2n], \quad n \in \mathbb{Z}.
\]
We then expand the signal \( x_1[n] \) by a factor of 3, resulting in the signal
\[
x_2[n] = \begin{cases} x_1[n/3] & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{otherwise} \end{cases}.
\]
Finally, we downsample \( x_2[n] \) by a factor of 2, resulting in the signal
\[
y[n] = x_2[2n], \quad n \in \mathbb{Z}.
\]
Find the most general conditions you can on the discrete time Fourier transform of the original signal \( x[n] \) under which you are guaranteed that it can be uniquely recovered from the signal \( y[n] \).

7. (3 + 9 points)
Let \( y(t) \) be a continuous time signal which is time limited to \((-3T, 3T)\).
Let
\[
x(t) = \sum_{n=-\infty}^{\infty} y(t - nT).
\]
(a) Verify that \( x(t) \) is well defined and is periodic with period \( T \).
(b) Let $a_n$ denote the Fourier series coefficient of $x(t)$, whose period is taken to be $T$. Express the $a_n$ in terms of the Fourier series $Y(j\omega)$ of $y(t)$.

8. (2 + 3 + 3 + 1 + 3 + 3 points)

Consider the continuous time signal

$$x(t) = u(t + 1.5) + u(t + 0.5) - u(t - 0.5) - u(t - 1.5).$$

(a) Sketch $x(t)$.
(b) Determine the Fourier transform of $x(t)$, i.e. $X(j\omega)$.
(c) Let $T = 1/3$. Let

$$y[n] = x(nT), \quad n \in \mathbb{Z}.$$  

Thus the discrete time signal $y[n]$ is the sequence of samples of $x(t)$ with sampling period $T = \frac{1}{3}$.
Let $Y(z)$ denote the z-transform of $y[n]$. Determine $Y(z)$.
(d) Determine the discrete time Fourier transform of $y[n]$, i.e. $Y(e^{j\omega})$.
(e) Let $h(t)$ denote the signal

$$h(t) = u(t) - u(t - \frac{1}{3}).$$

Let

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} y[n]h(t - nT).$$

Thus $\tilde{x}(t)$ can be thought of as the output of a zero order hold with impulse response $h(t)$, whose input is the signal $\sum_{n=-\infty}^{\infty} y[n]\delta(t - nT)$.
Determine the Fourier transform of $\tilde{x}(t)$, i.e. $\tilde{X}(j\omega)$.
(f) Determine $\int_{-\infty}^{\infty} |x(t) - \tilde{x}(t)|^2 \, dt$.

9. (2 + 2 + 5 + 3 + 3 points)

Consider the rational Laplace transform

$$R(s) = \frac{s - 1}{(s^2 + 2s + 2)(s^2 + 2s + 5)}.$$  

This problem asks some questions about the root locus plot for this rational Laplace transform, i.e. about the locations of the zeros of $1 + KR(s)$ as $K$ ranges over real values. It is not necessary for you to determine the entire root locus in order to answer the questions in this problem.
In each part of the problem asking for an explanation, at most 1 point will be given for a correct answer without a correct explanation.

(a) Does the value $s_0 = 0$ lie on the root locus for some value $K > 0$? Explain what criterion you used in arriving at your answer.
(b) Does the value $s_0 = 0$ lie on the root locus for some value $K < 0$? Explain what criterion you used in arriving at your answer.

(c) Clearly state and explain the angle criterion for determining whether a complex number $s_0$ can be on the root locus plot for $R(s)$.

(d) Does the value $s_0 = 1 + j\frac{3}{2}$ lie on the root locus for some value $K > 0$? Explain how you arrived at your answer.

(e) Does the value $s_0 = j100$ lie on the root locus for some value $K > 0$? Explain how you arrived at your answer.