EECS 120 **Fall 2004**

Sample Final Solutions

1. False. If a causal LTI system has any poles in the right half plane then it will be unstable.

2. True. We need to verify that $\int_{-\infty}^{\infty} |X(j\omega)+Y(j\omega)|^2 \, d\omega \geq \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$

$$\int_{-\infty}^{\infty} (X(j\omega)+Y(j\omega))(X^*(j\omega)+Y^*(j\omega))d\omega \geq \int_{-\infty}^{\infty} X(j\omega)X^*(j\omega)d\omega$$

$$\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega + \int_{-\infty}^{\infty} X^*(j\omega)Y(j\omega)d\omega + \int_{-\infty}^{\infty} Y(j\omega)Y^*(j\omega)d\omega \geq 0$$

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt + \int_{-\infty}^{\infty} x^*(t)y(t)dt + \int_{-\infty}^{\infty} y(t)y^*(t)dt \geq 0 \quad (1)$$

Where the last line follows from Parseval's Theorem. The statement is true, since the evaluations of the integrals in equation (1) will be non-negative. This is true since it is given that x(t) ≥ 0, and y(t) ≥ 0.

3.

(a) $Y(j\omega) = \frac{1}{2\pi}\int X(j\theta)Q(j(\omega-\theta))d\theta$

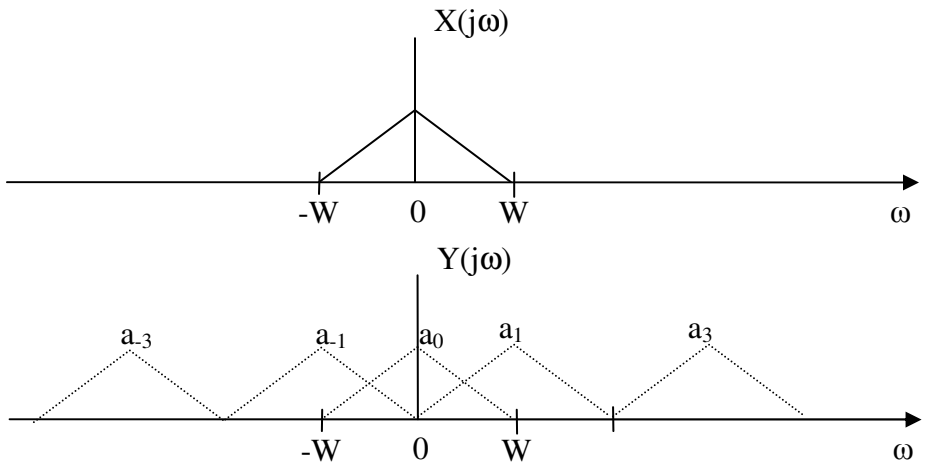
$Q(j\omega) = 2\pi\sum_k a_k \delta(\omega - k\omega_s)$, where $\omega_s = \frac{2\pi}{T}$, and

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} (\delta(t) + \delta(t-\frac{T}{4}))e^{-jk\frac{2\pi}{T}t}dt$$

$$a_k = \frac{1}{T}[1+e^{-jk\frac{\pi}{2}}]$$

So we have that $Y(j\omega) = \frac{1}{2\pi}\int X(j\theta)Q(j(\omega-\theta))d\theta = \sum_k a_k X(\omega - k\omega_s)$

(b) Consider the condition $\frac{2\pi}{T} = W$, $Y(j\omega)$ is sketched below for this case:



Under normal impulse train sampling condtions, we would need to sample at the Nyquist rate to perfectly recover x(t) from y(t). However, we observe that $a_{2+4k} = 0$, for all integers k. Thus as can be seen in the figure, we can sample at $\omega_s = W$, and still recover x(t). This would be done by bandpass filtering y(t) with a filter with pass band [$W < |\omega| < 2W$] and gain $1/a_1$. We can then multiply this resulting signal by cos(Wt) and low pass filter with cutoff $\omega = W$ and gain 2 to recover x(t).

4. Taking the Laplace transform of the differential equation we get:

$$s^2Y(s) - Y(s) = sX(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^2 - 1} = \frac{0.5}{s+1} + \frac{0.5}{s-1}$$

If we take the ROC to be Re{s} < -1, then this LTI system will be anticausal.

If we took the inverse Laplace transform we would find that this system has an impulse response of the form:

$$h(t) = 0.5e^{-t}u(-t) + 0.5e^{t}u(-t)$$

5. x[n] = $\delta[n]-\delta[n-1]+\frac{1}{4}\delta[n-2]$

X(z) = $1-z^{-1}+\frac{1}{4}z^{-2}$

$$y[n]=\frac{1}{2^{n-2}}u[n-2]$$

Y(z) = $\frac{z^{-1}}{z-\frac{1}{2}}$, ROC |z| > $\frac{1}{2}$

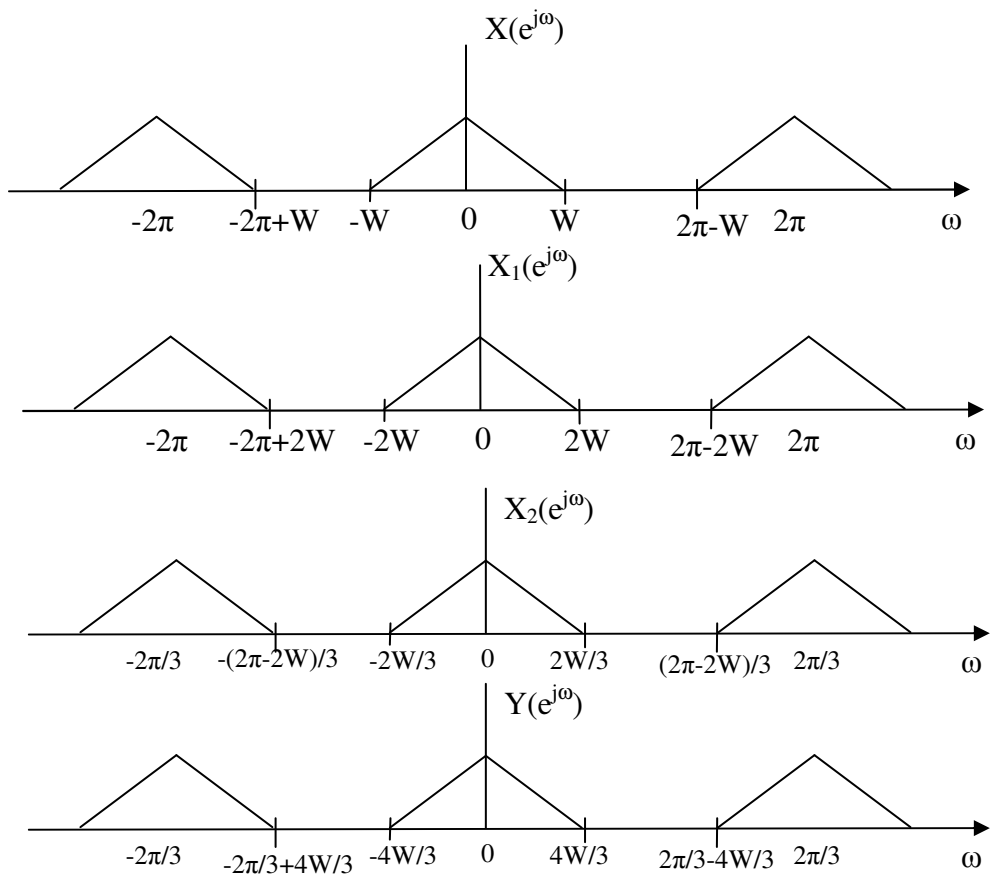
$$H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}}{(1-z^{-1}+\frac{1}{4}z^{-2})(z-\frac{1}{2})}=\frac{z}{(z^2-z+\frac{1}{4})(z-\frac{1}{2})}$$

This system has 3 poles at z = ½.

We know that Y(z)=X(z)H(z), and also that the ROC of Y(z) must contain the ROC of the intersection of X(z) with H(z). Since the ROC of X(z) is the whole complex plane, the ROC of Y(z) must contain the ROC of H(z). Thus the ROC of H(z) must be |z| > ½.

Since the ROC of H(z) is |z| > ½, it contains the unit circle and thus the system must be stable.

6. Sketch:



To avoid aliasing going from x[n] to x_1[n] we must have:

$$2\pi - 2W > 2W \text{ , => } W < \frac{\pi}{2}$$

To avoid aliasing going from x_2[n] to y[n] we must have:

$$\frac{2\pi}{3} - \frac{4W}{3} > \frac{4W}{3}, \; W < \frac{\pi}{4},$$

Thus we can uniquely recover x[n] from y[n] if we have

$$X(e^{j\omega}) = 0 \text{ for } |\omega| > \frac{\pi}{4}$$

7. $x(t) = \sum_{n=-\infty}^{\infty} y(t-nT)$, y(t) is time limited to (-3T, 3T)

(a) $x(t-T) = \sum_{n=-\infty}^{\infty} y(t-T-nT) = \sum_{n=-\infty}^{\infty} y(t-(n+1)T) = \sum_{m=-\infty}^{\infty} y(t-mT) =$ x(t)

Thus, x(t) is periodic with period T. Also, since y(t) is time limited, the summation is finite for all t and thus x(t) is well defined.

(b) $a_n = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{l=-\infty}^{\infty} y(t-lT)e^{-jn\frac{2\pi}{T}t}\,dt$

$$a_n = \underset{M\to\infty}{LIM}\frac{1}{MT}\int_{-MT/2}^{MT/2} \sum_{l=-\infty}^{\infty} y(t-lT)e^{-jn\frac{2\pi}{T}t}\,dt \qquad (1)$$

where the second step comes from the fact that the integral of x(t) over any interval T gives the same number. In fact, there is no need to take a limit in the last step. We have equality for each M > 0.

Now, we notice that since y(t) is time limited to (-3T,3T) in the integrand of the integral on the right hand side of the last equation, each of the terms for $-\frac{MT}{2} \le (l-3)T < (l+3)T \le \frac{MT}{2}$ can be replaced by the term:

$$\int_{-\infty}^{\infty} y(t-lT)e^{-jn\frac{2\pi}{T}t}\,dt$$

Note that there are (M-5) such terms.
This integral, can in turn be simplified as follows:

$$\int_{-\infty}^{\infty} y(t-lT)e^{-jn\frac{2\pi}{T}t}\,dt = \int_{-\infty}^{\infty} y(s)e^{-jn\frac{2\pi}{T}(s+lT)}\,dt$$

$$= \int_{-\infty}^{\infty} y(s)e^{-jn\frac{2\pi}{T}s}\,dt$$

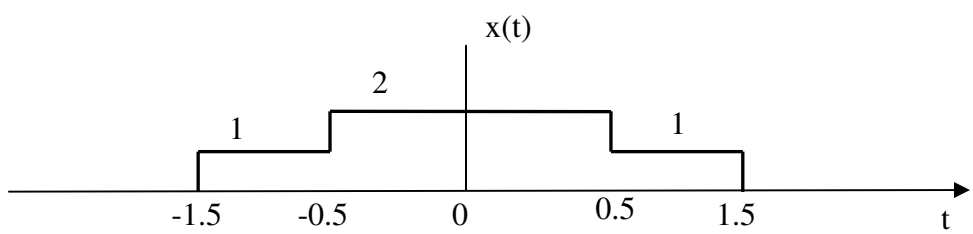
$$= Y(j\frac{2\pi}{T}n)$$

Finally, we observe that the contributions from the terms on the right hand side of the equation (1) that have not been taken care of is bounded independent of M (there are at most 10 terms that could be non-zero), and so finally we get:

$$a_n = \underset{M \to \infty}{LIM} \frac{M-5}{MT} Y(j\frac{2\pi}{T}n)$$

$$a_n = \frac{1}{T} Y(j\frac{2\pi}{T}n)$$

8.
(a)



(b) x(t) = rect(3t) + rect(1t),

where rect(At) = $\begin{cases} 1, & for \quad |t| < A/2 \\ 0, & else \end{cases}$

$$X(j\omega) = \frac{2\sin(\omega\frac{3}{2})}{\omega} + \frac{2\sin(\omega\frac{1}{2})}{\omega}$$

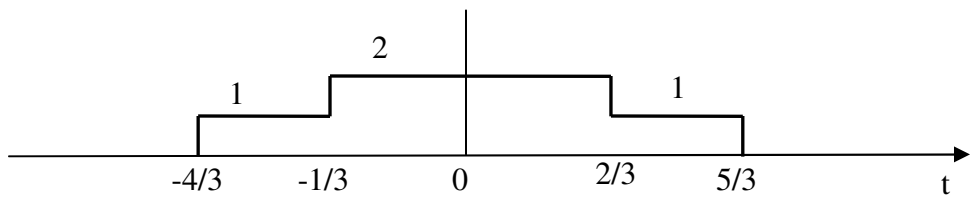
(c) y[n] = x(nT) = δ[n+4]+ δ[n+3]+ δ[n+2]+ 2δ[n+1]+ 2δ[n]+ 2δ[n-1]+ δ[n-2]+ δ[n-3]+ δ[n-4] =

$Y(z) = z^4 + z^3 + z^2 + 2z^1 + 2 + 2z^{-1} + z^{-2} + z^{-3} + z^{-4}$

(d) $Y(e^{j\omega}) = Y(z)|\ z=e^{j\omega}$

$= e^{j4\omega} + e^{j3\omega} + e^{j2\omega} + 2\,e^{j\omega} + 2 + 2e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega}$

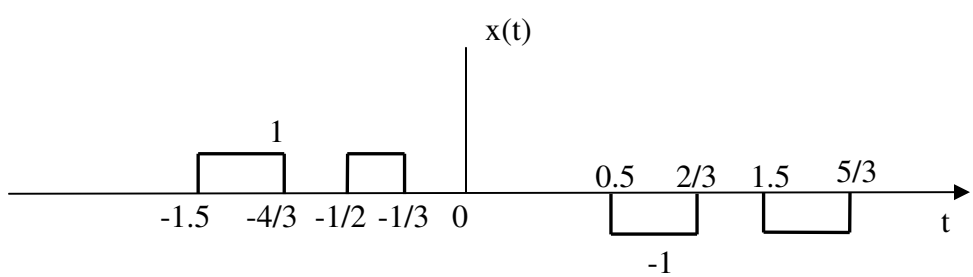
(e) Sketch of $\tilde{x}(t)$:



$$\tilde{x}(t) = rect(3(t - \frac{1}{6})) + rect(t - \frac{1}{6})$$

$$X(\tilde{j\omega}) = (\frac{2\sin(\omega\frac{3}{2})}{\omega} + \frac{2\sin(\omega\frac{1}{2})}{\omega})e^{-j\omega\frac{1}{6}}$$

(f) Sketch of $x(t) - \tilde{x}(t)$



$$\int_{-\infty}^{\infty} |\, x(t) - \tilde{x}(t)\,|^2\, dt = 1/6+1/6+1/6+1/6 = 2/3$$

9.

(a) $1 + KR(s) = 1 + \dfrac{K(s-1)}{(s^2+2s+2)(s^2+2s+5)}$

1+KR(0) = 1-K/10 = 0, K = 10

Yes, for K = 10 $s_0 = 0$ lies on the root locus.

Also, we know that the angle criterion states that $\angle R(s_0 = 0)$ must be equal to π for s_0 to lie on the root locus for K > 0. R(0) = -1/10, thus this condition is satisfied.

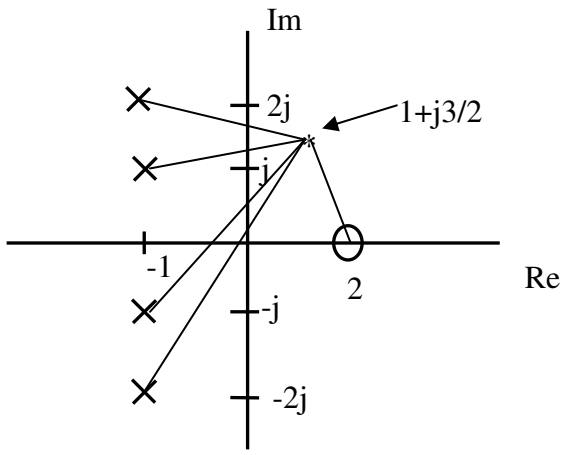
(b) No, we explicitly found that it only lies on the root locus for K = 10 in part (a).

Also, the phase of R(s) at s = 0 was found to be π from part (a). For K < 0, we would need the phase to be 2π for s=0 to lie on the root locus by the angle criterion.

(c) Since we know that $R(s_0) = -1/K$ must be satisfied for points on the root locus. The anlge criterion states that $\angle R(s_0)$ must be an odd multiple of π for s_0 to be on the root locus for K > 0, and an even multiple of π for s_0 to be on the root locus for K < 0.

(d) Poles of R(s): s = -1±j, -1±2j
Zeros of R(s): s = 1

Pole-zero plot of R(s):



For s_0 = 1+j3/2, we can calculate the phase of $R(s_0)$ using the vector sum approach to be:

123.69 – (60.255+51.34+14.04-14.04) = 12.1°

Since the phase of $R(s_0)$ is not a multiple of π, s_0 cannot lie on the root locus.

Since calculators are not allowed on the exam, we can eyeball the pole zero plot and estimate the phase. We would do this by noting that by symmetry the phase contributions from the poles at -1+j, -1+2j cancels out. The phase contributions from the poles at -1-j, -1-2j is approximately -2(90- θ), where θ = arctan(2/3). The phase contribution from the zero is 90 + θ. Since tan(30) = $1/\sqrt{3} < 2/3$, one has $\theta > 30°$. So one expects that the net angle should be greater than 0.

(e) For s_0 = j100, we can approximate the phase to be composed of a 90 degrees contribution from the zero, and -360 degree contribution from the poles, giving us a net phase of -270 degrees. Since the phase is not 180 degrees, the point clearly does not lie on the root locus.