Sample Midterm 2

• The exam is for one hour and 50 minutes.

• The maximum score is 100 points. The maximum score for each part of each problem is indicated.

• The exam is closed-book and closed-notes. Calculators, computing, and communication devices are NOT permitted.

• Four double-sided sheets of notes are allowed. These should be legible to normal eyesight, i.e. the lettering should not be excessively small.

• No form of collaboration between students is allowed.

1. (10 points) State whether the following are true or false. In each case, give a brief explanation. A correct answer with a correct explanation gets 5 points. A correct answer without a correct explanation gets 2 points. A wrong answer gets 0 points.

   (a) Frequency modulation satisfies the superposition principle, i.e. for \( i = 1, 2 \) if \( u_i(t) \) denotes the signal resulting from frequency modulating \( m_i(t) \) onto the carrier \( A_c \cos(\omega_c t) \) then \( au_1(t) + bu_2(t) \) is the signal resulting from frequency modulation \( am_1(t) + bm_2(t) \) onto the carrier.

   (b) If a signal \( x(t) \) has Fourier transform \( X(j\omega) \) such that
   \[
   X(j\omega) = 0 \text{ if } |\omega| \leq 2\omega_M
   \]
   and
   \[
   X(j\omega) = 0 \text{ if } |\omega| \geq 3\omega_M
   \]
   then \( x(t) \) can be recovered from the sequence of its samples at the times \( nT, n \in \mathbb{Z} \), where \( \frac{2\pi}{T} = 3\omega_M \).

2. (10 points)
   Let \( \omega_0 << \omega_c \). The modulating signal
   \[
   m(t) = \cos(\omega_0 t) - 0.7 \sin(2\omega_0 t)
   \]
   is USSB amplitude modulated onto the carrier \( A_c \cos(\omega_c t) \). Find an explicit expression for the modulated signal \( u(t) \).

   \textit{Hint} : First draw a picture of the Fourier transform of \( u(t) \).

3. (10 points)
A lowpass signal \( m(t) \) which is bandlimited to \((-\omega_M, \omega_M)\) is modulated onto the carrier \( A_c \cos(\omega_c t) \) using DSB-SC amplitude modulation. Here \( \omega_M << \omega_c \). The resulting signal is modulated onto the carrier \( A_d \cos(\omega_d t) \), where \( \omega_c << \omega_d \), using USSB amplitude modulation. Let \( x(t) \) denote the resulting signal.

Assume that the Fourier transform of \( m(t) \) is

\[
M(j\omega) = \begin{cases} 
e e^{j\omega} & \text{if } |\omega| < \omega_M \\
0 & \text{otherwise} \end{cases}
\]

Sketch the Fourier transform \( X(j\omega) \). Note: Sketch both the magnitude and the phase of the Fourier transform.

4. \((5 + 10 + 3 + 10 + 2 \text{ points})\)

For a linear time invariant system, it is known that the system function is given by

\[
H(s) = \frac{6(s - 4)}{(s + 3)(s^2 - 4s + 13)}
\]

(a) Draw the pole-zero diagram for \( H(s) \).

(b) Suppose that you are told that the system is causal. Find the impulse response \( h(t) \) of the LTI system.

(c) Suppose you are told that the system is causal. Is it stable? Explain your answer.

(d) Suppose that you are told that the system is stable. Find the impulse response \( h(t) \) of the LTI system.

(e) Suppose you are told that the system is stable. Is it causal? Explain your answer.

5. \((10 \text{ points})\)

Let \( x(t) \) and \( y(t) \) be signals, each of which is bandlimited to \((-\omega_M, \omega_M)\). Let \( a > 1 \) be a real number and let

\[
z(t) = \cos(a \omega_M t) + y(t).
\]

Let \( v(t) = x(t)z(t) \). The signal \( v(t) \) is periodically sampled with period \( T \).

For what values of \( a \) and \( T \) is it possible to recover both \( x(t) \) and \( y(t) \) from the samples \( v(nT), n \in \mathbb{Z} \) ?

6. \((10 \text{ points})\)

Let \( x[n] \) be a discrete time signal, with discrete time Fourier transform \( X(e^{j\omega}) \). Recall that

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \quad \text{and}
\]

\[
x[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(e^{j\omega})e^{j\omega n}d\omega.
\]

Assume that

\[
X(e^{j\omega}) = 0, \quad \text{if } \omega_M < |\omega| \leq \pi.
\]
Let $H(e^{j\omega})$ denote the ideal low pass filter
\[
H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{\omega - 2\pi n}{2\omega_0}\right),
\]
where $\Pi(x)$ equals 1 if $|x| < \frac{1}{2}$ and equals 0 otherwise. We assume here that $\omega_0 < \pi$.

Let $y[n]$ denote the result of passing $x[n]$ through this filter. Let $y_b[n]$ be the decimation of $y[n]$ to even times, i.e.
\[
y_b[n] = y[2n], \quad n \in \mathbb{Z}.
\]

Let $x_b[n]$ denote the decimation of $x[n]$ to even times, i.e.
\[
x_b[n] = x[2n], \quad n \in \mathbb{Z},
\]
and let $z[n]$ denote the result of passing $x_b[n]$ through the filter.

Find the conditions on $\omega_M$ and $\omega_0$ under which $z[n] = y_b[n]$ for all $n \in \mathbb{Z}$.

7. (5 points)
Let $x_1(t)$ have Laplace transform
\[
X_1(s) = \frac{s - 1}{s + 2}, \quad \text{Re}(s) > -2,
\]
and let $x_2(t)$ have Laplace transform
\[
X_2(s) = \frac{s + 2}{s^2 + 1}, \quad \text{Re}(s) < 0.
\]
Let
\[
x_3(t) = x_1(t) * x_2(t),
\]
denote the convolution of $x_1(t)$ and $x_2(t)$. Find the Laplace transform $X_3(s)$ of $x_3(t)$ (together with its region of convergence!).

8. (15 points)
In the following problem, on the next page, every correct answer with a correct reason gets 3 points, every correct answer with a wrong reason gets 1 point, and every wrong answer gets 0 points.
Match the pole/zero plots (a)-(e) with the corresponding magnitude responses (1)-(5). In each case, provide a brief justification.