EECS 120 **Fall 2004**

Homework 5 Solutions

(1) Problem 5.47 of OWN:

(a) $X(e^{j\omega}) = X(e^{j(\omega-1)})$. Taking the Inverse Fourier transform of the previous equation, we get $x[n] = e^{jn}x[n]$ for all n, which implies $x[n] = 0$ for $|n| > 0$. Therefore the statement is **True**.

(b) $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$. Again, taking the Inverse Fourier transform we obtain $x[n] = e^{j\pi n}x[n] = (-1)^n x[n]$. This only implies that, $x[n] = 0$ for all odd integers n; it says nothing about x[n] for n even. Hence the given assertion is **False**.

(c) $X(e^{j\omega}) = X(e^{j\omega/2})$. We claim that, if $X(e^{j\omega})$ is continuous at $\omega = 0$, the only possible $X(e^{j\omega})$ which satisfies this relation is $X(e^{j\omega}) = constant$. In order to prove this, let $\omega_1 \neq \omega_2$. Then by the given relation we have

$$X(e^{j\omega_1}) = X(e^{j\omega_1/2^k})$$

for all k. Similarly, we have

$$X(e^{j\omega_2}) = X(e^{j\omega_2/2^k})$$

Now, assuming that the Fourier transform is continuous at $\omega = 0$ we have $X(e^{j\omega_1/2^k}) \to X(e^{j0})$ as $k \to \infty$. Similarly, $X(e^{j\omega_2/2^k}) \to X(e^{j0})$ as $k \to \infty$. Therefore, we must have $X(e^{j\omega_1}) = X(e^{j\omega_2})$ and thus $X(e^{j\omega}) = constant$. This implies that $x[n] = constant\ \delta[n]$. Hence $x[n] = 0$ for $|n| > 0$. Therefore, the given statement is **True**.

(d) $X(e^{j\omega}) = X(e^{j2\omega})$. Substituting $\omega = \omega/2$ in this equation gives back part the same relation in part (c). Similarly, we can get this relation ship from the one in part (c) by substituting $\omega = 2\omega$. Therefore this statement is also **True**.

(2) Problem 5.48 of OWN:

(a) The system is specified by the following pair of difference equations.

$$\begin{aligned} y[n] + \frac{1}{4}y[n-1] + w[n] + \frac{1}{2}w[n-1] &= \frac{2}{3}x[n] \\ y[n] - \frac{5}{4}y[n-1] + 2w[n] - 2w[n-1] &= -\frac{5}{3}x[n] \end{aligned}$$

Taking the Fourier transform of this pair of equations, we get

$$\begin{aligned} Y(e^{j\omega}) + \frac{1}{4}Y(e^{j\omega})e^{-j\omega} + W(e^{j\omega}) + \frac{1}{2}W(e^{j\omega})e^{-j\omega} &= \frac{2}{3}X(e^{j\omega}) \\ Y(e^{j\omega}) - \frac{5}{4}Y(e^{j\omega})e^{-j\omega} + 2W(e^{j\omega}) - 2W(e^{j\omega})e^{-j\omega} &= -\frac{5}{3}X(e^{j\omega}) \end{aligned}$$

Eliminating $W(e^{j\omega})$ from these equations, we get

$$\begin{aligned} \frac{\left[1+\frac{1}{4}e^{-j\omega}\right]Y(e^{j\omega}) - \frac{2}{3}X(e^{j\omega})}{1+\frac{1}{2}e^{-j\omega}} &= \frac{\left[1-\frac{5}{4}e^{-j\omega}\right]Y(e^{j\omega}) + \frac{5}{3}X(e^{j\omega})}{2-2e^{-j\omega}} \\ \Rightarrow \left[1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}\right]Y(e^{j\omega}) &= \left[3-\frac{1}{2}e^{-j\omega}\right]X(e^{j\omega}) \qquad (1) \\ \Rightarrow \left[1-\frac{1}{4}e^{-j\omega}\right]\left[1-\frac{1}{2}e^{-j\omega}\right]Y(e^{j\omega}) &= \left[3-\frac{1}{2}e^{-j\omega}\right]X(e^{j\omega}) \\ \Rightarrow H(e^{j\omega}) &= \frac{\left[3-\frac{1}{2}e^{-j\omega}\right]}{\left[1-\frac{1}{4}e^{-j\omega}\right]\left[1-\frac{1}{2}e^{-j\omega}\right]} \qquad (2) \end{aligned}$$

In order to obtain the impulse response of the system, we first write (2) as a partial fraction expansion to give

$$H(e^{j\omega}) = -\frac{1}{\left[1-\frac{1}{4}e^{-j\omega}\right]} + \frac{4}{\left[1-\frac{1}{2}e^{-j\omega}\right]}$$

Taking the Inverse Fourier transform of this equation we get,

$$h[n] = \left[-\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{2}\right)^n\right]u[n]$$

(b) Taking the Inverse Fourier transform of (1) we get the following difference equation between $x[n]$ and $y[n]$.

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 3x[n] - \frac{1}{2}x[n-1]$$

(3) Problem 5.50 of OWN:

(a) (i) The input to a LTI system is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

and the corresponding output is

$$y[n] = \left(\frac{1}{3}\right)^n u[n]$$

Taking the Fourier transform of both the input and the output we evaluate the frequency response of the system to be

$$\begin{aligned} H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} &= \frac{\frac{1}{1-\frac{1}{3}e^{-j\omega}}}{\frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}} \\ &= \frac{1-\frac{1}{2}e^{-j\omega}}{\left[1-\frac{1}{3}e^{-j\omega}\right]\left[1-\frac{1}{4}e^{-j\omega}\right]} \\ &= \frac{-2}{1-\frac{1}{3}e^{-j\omega}} + \frac{3}{1-\frac{1}{4}e^{-j\omega}} \end{aligned} \tag{3}$$

Taking the Inverse Fourier transform of the last equation, we get the following impulse response for the system.

$$h[n] = \left[3\left(\frac{1}{4}\right)^n - 2\left(\frac{1}{3}\right)^n\right]u[n]$$

(ii) Cross multiplying terms in (3) we get

$$\left[1-\frac{1}{3}e^{-j\omega}\right]\left[1-\frac{1}{4}e^{-j\omega}\right]Y(e^{j\omega}) = \left[1-\frac{1}{2}e^{-j\omega}\right]X(e^{j\omega})$$

Taking the Inverse Fourier transform of the above equation, we get

$$y[n] - \frac{7}{12}y[n-1] + \frac{1}{12}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

(b) Taking the Fourier transform of the given input and its corresponding output, we get

$$\begin{aligned} H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} &= \frac{\frac{1}{1-\frac{1}{4}e^{-j\omega}}}{\frac{2-\frac{1}{2}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2}} \\ &= \frac{(1-\frac{1}{2}e^{-j\omega})^2}{(1-\frac{1}{4}e^{-j\omega})(2-\frac{1}{2}e^{-j\omega})} \end{aligned}$$

Therefore, if the output of this system is $\delta[n] - \left(\frac{1}{2}\right)^n u[n]$, then the input is given by

$$\begin{aligned} X(e^{j\omega}) &= \frac{Y(e^{j\omega})}{H(e^{j\omega})} \\ &= \frac{\frac{\frac{1}{2}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}}{H(e^{j\omega})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})(2-\frac{1}{2}e^{-j\omega})}{(1-\frac{1}{2}e^{-j\omega})^2(1+\frac{1}{2}e^{-j\omega})} \\
&= \frac{1}{2} - \frac{\frac{9}{8}}{1+\frac{1}{2}e^{-j\omega}} + \frac{\frac{3}{8}}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{4}}{(1-\frac{1}{2}e^{-j\omega})^2}
\end{aligned}
$$

Taking the inverse Fourier transform of the last equation, we get

$$
x[n] = \frac{1}{2}\delta[n] - \left[\frac{9}{8}\left(-\frac{1}{2}\right)^n - \frac{3}{8}\left(\frac{1}{2}\right)^n - \frac{1}{4}(n+1)\left(\frac{1}{2}\right)^n\right]u[n]
$$

(4) Problem 5.53 of OWN:

(a) By definition of the Fourier transform, we have

$$
\begin{aligned}
\frac{1}{N}X(e^{j\omega}) &= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \frac{1}{N}\sum_{n=0}^{N_1-1} x[n]e^{-j\omega n} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\omega n}
\end{aligned}
$$

where, step 2 follows from the fact that $x[n]$ is non-zero only in the interval $[0, N_1]$ and step 3 follows from the fact that $N \geq N_1$ and $x[n] = 0$ for $n > N_1$. Substituting $\omega = \frac{2\pi k}{N}$ in the previous equation we get,

$$
\begin{aligned}
\frac{1}{N}X(e^{j(2\pi k/N)}) &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(2\pi k/N)n} \\
&= \tilde{X}[k]
\end{aligned}
$$

(b) From the given signals $x_1[n]$ and $x_2[n]$ form two corresponding periodic signals $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ with period 4, in the following manner

$$
\begin{aligned}
\tilde{x}_1[n] &= \sum_{l=\infty}^{\infty} x_1[n-4l] \\
\tilde{x}_2[n] &= \sum_{l=\infty}^{\infty} x_2[n-4l]
\end{aligned}
$$

It is clear that both these new signals are periodic with period 4. Since $x_1[n]$ is non-zero only for $0 \leq n \leq 3$, it follows that $\tilde{x}_1[n] = x_1[n]$ in this interval.

However, $\tilde{x}_2[n] = x_2[n] + x_2[n-4] + x_3[n+4]$ for $0 \leq n \leq 3$ because $x_2[n]$ is non-zero only in the interval $-2 \leq n \leq 7$. Convince yourself by verification that $\tilde{x}_1[n] = \tilde{x}_2[n]$. Therefore, they must also have the same Fourier series coefficients, i.e.,

$$
\begin{aligned}
\frac{1}{4}\sum_{n=0}^{3} \tilde{x}_1[n]e^{-j(2\pi k/N)n} &= \frac{1}{4}\sum_{n=0}^{3} \tilde{x}_2[n]e^{-j(2\pi k/N)n} \\
\Rightarrow \sum_{n=0}^{3} x_1[n]e^{-j(2\pi k/N)n} &= \sum_{n=0}^{3} \tilde{x}_2[n]e^{-j(2\pi k/N)n} \\
\Rightarrow \sum_{n=0}^{3} x_1[n]e^{-j(2\pi k/N)n} &= \sum_{n=0}^{3} (x_2[n] + x_2[n-4] + x_3[n+4])e^{-j(2\pi k/N)n} \\
&= \sum_{n=0}^{3} x_2[n]e^{-j(2\pi k/N)n} + \sum_{n=0}^{3} x_2[n+4]e^{-j(2\pi k/N)n} \\
&\quad + \sum_{n=0}^{3} x_2[n-4]e^{-j(2\pi k/N)n} \\
&= \sum_{n=0}^{3} x_2[n]e^{-j(2\pi k/N)n} + \sum_{n=-4}^{-1} x_2[n]e^{-j(2\pi k/N)(n-4)} \\
&\quad + \sum_{n=4}^{7} x_2[n]e^{-j(2\pi k/N)(n+4)} \\
&= \sum_{n=0}^{3} x_2[n]e^{-j(2\pi k/N)n} + \sum_{n=-4}^{-1} x_2[n]e^{-j(2\pi k/N)n} \\
&\quad + \sum_{n=4}^{7} x_2[n]e^{-j(2\pi k/N)n} \\
&= \sum_{n=-4}^{7} x_2[n]e^{-j(2\pi k/N)n} \\
&= \sum_{n=-2}^{7} x_2[n]e^{-j(2\pi k/N)n} \\
\Rightarrow X_1(e^{j(2\pi k/4)}) &= X_2(e^{j(2\pi k/4)})
\end{aligned}
$$

Hence proved.

(5) Matlab problem:

(d) From Fig. 6 is it clear that $y_2[n]$ looks more like the input $x[n]$. To understand this behavior, we need to look at the frequency response of both the systems. From Fig. 1 and 3, we observe that both the systems have identical magnitude responses. So, the key aspect in which these systems differ is their phase response. From Fig. 2 we can observe that for system 1 the phase of low frequencies are affected rapidly whereas the high frequencies are not affected significantly. However, for system 2 the inverse happens, i.e., high frequencies are affected whereas the phase of the low frequencies remains more of less constant (see Fig. 4). Now the input to both these systems is $x[n] = (0.95)^n u[n]$, which is a low pass signal, i.e., most of the energy of the signal lies in the lower frequencies. Therefore, the output of the second system looks more like the input when compared to the output of the second system.

Fig. 7 shows the location of the poles and zeros of both the systems in the z-plane. You will learn later in the course how we can geometrically evaluate the phase of the system from its pole-zero plot. Intuitively, in order to evaluate the Fourier transform of the system at ω we have to evaluate the Z transform of the system at $z = e^{j\omega}$, i.e., as we move along the unit circle we can evaluate both the phase and the magnitude of the Fourier transform. Observe that there is a rapid change in the phase of the system whenever you pass in between the pole and zero of the system. For example, in system one when ω is close to 0 we pass in between the pole and zero and hence the phase changes rapidly. However, the same effect takes place in the second system at $\omega = \pi$ and hence the phase changes rapidly at π.

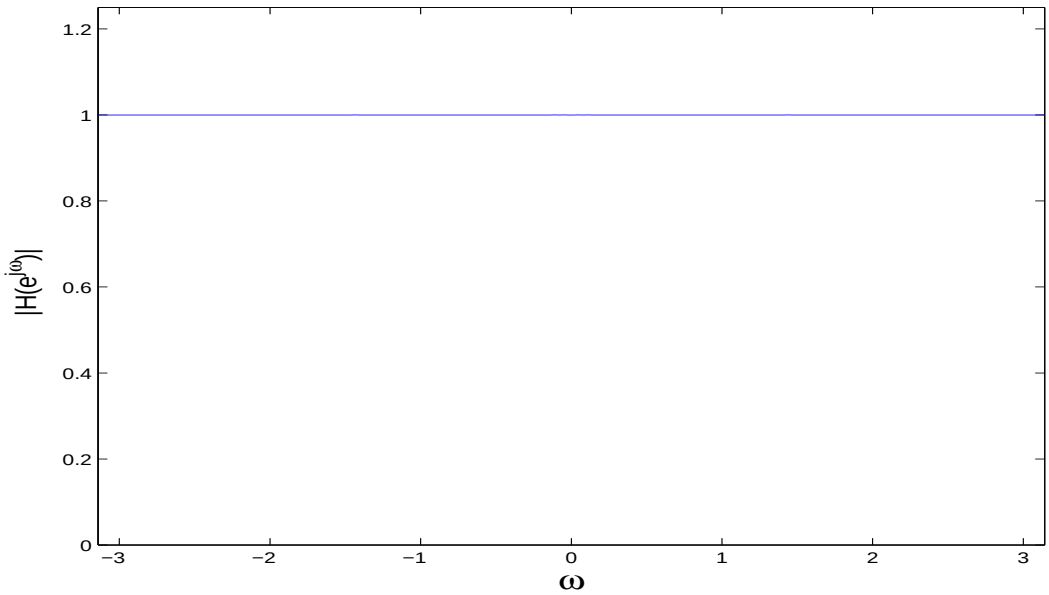

Figure 1: Magnitude plot of the first system

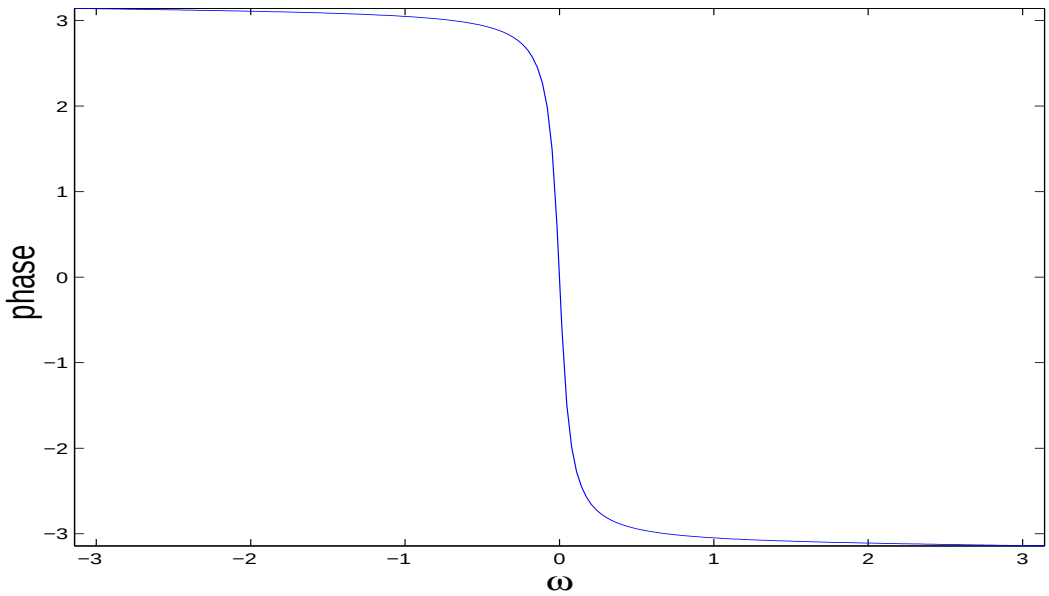

Figure 2: Phase plot of the first system

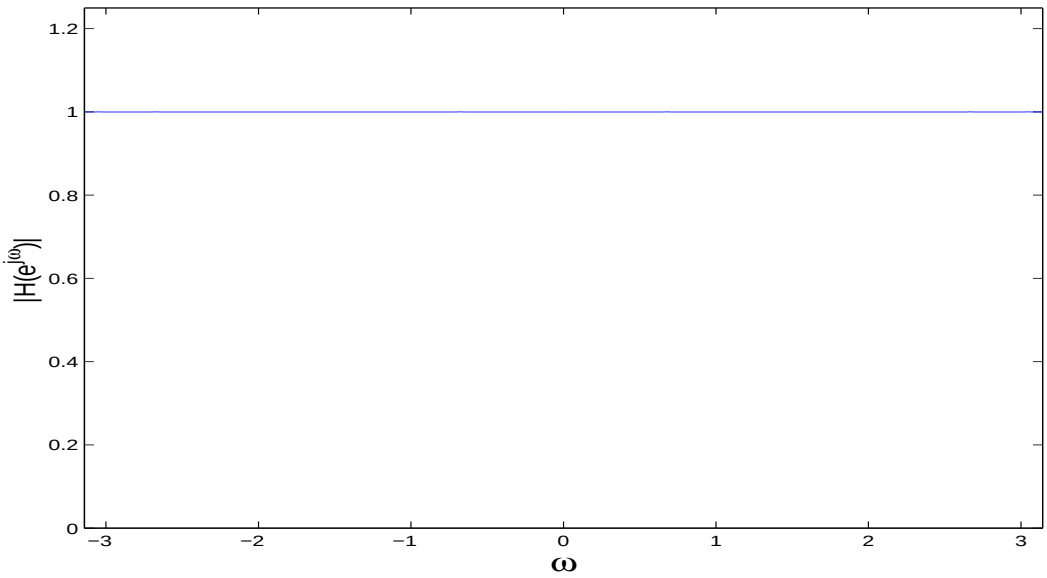

Figure 3: Magnitude plot of the second system

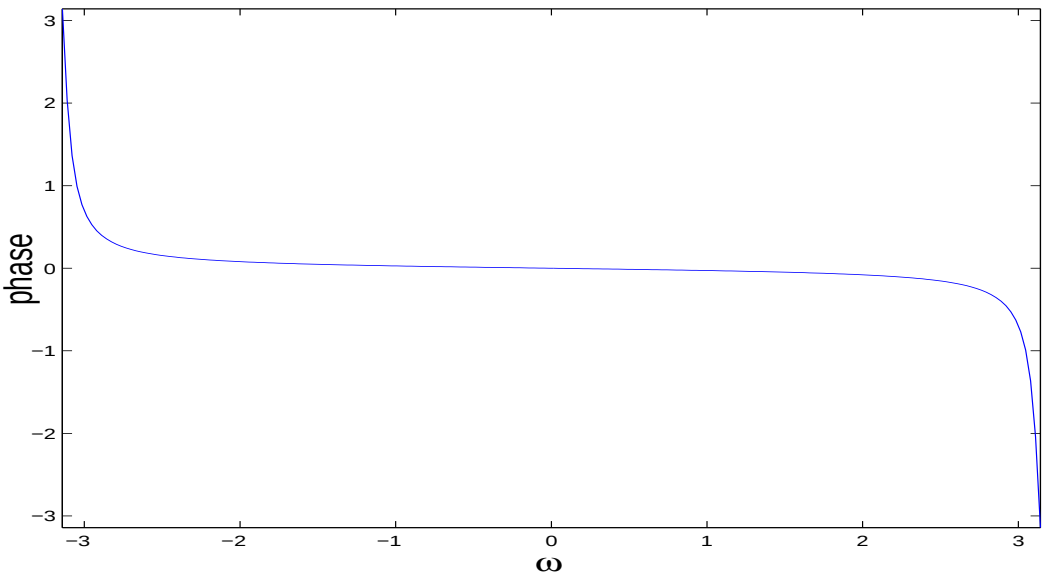

Figure 4: Phase plot of the second system

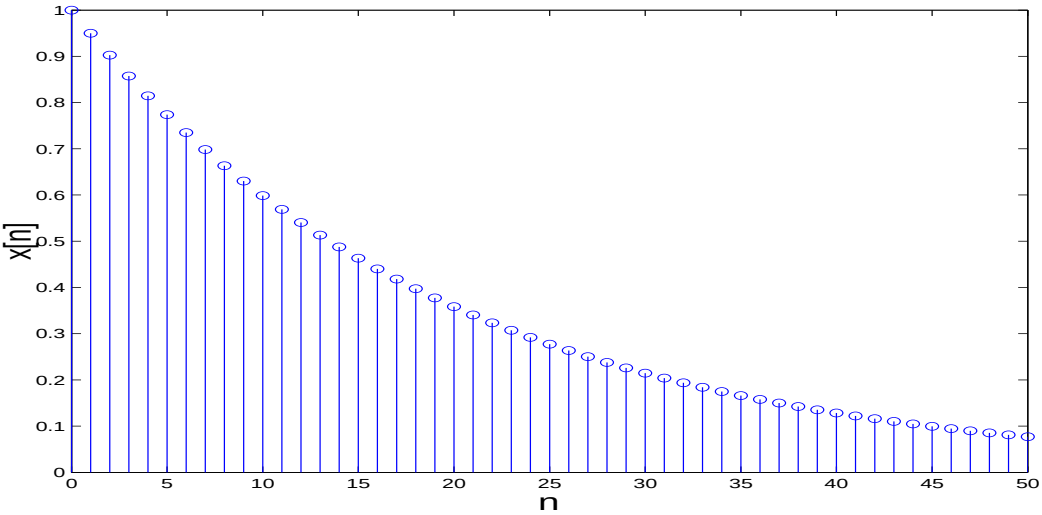

Figure 5: Stem plot of the input $x[n]$

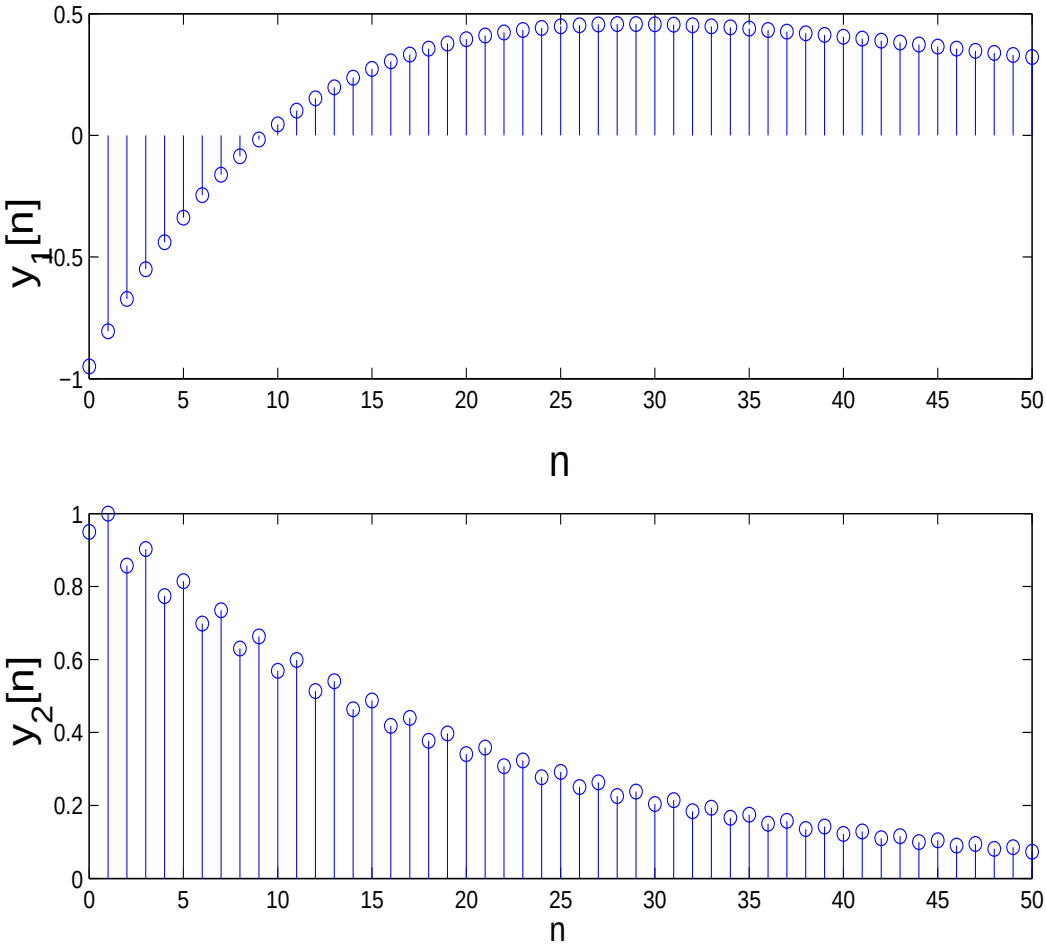

Figure 6: stem plots of the output of both the systems corresponding to the input $x[n]$

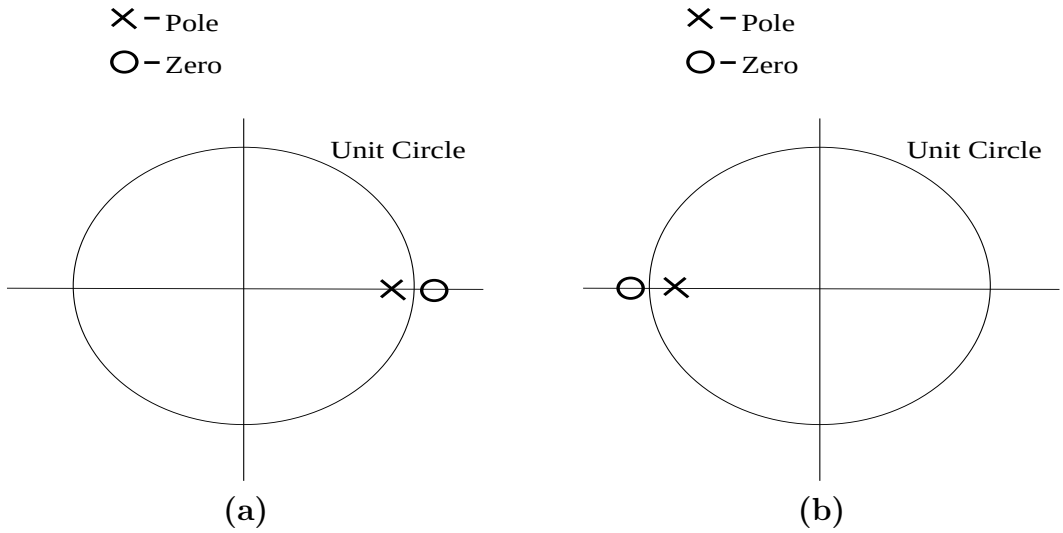

Figure 7: The pole-zero plot for both the systems is drawn in the z-plane.