Consider the constant coefficient difference equation :

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

The associated homogeneous equation is :

$$\sum_{k=0}^{N} a_k y[n-k] = 0 \ .$$

Consider the polynomial $p(z) = \sum_{k=0}^{N} a_k z^{N-k}$.
Factor it as

$$p(z) = a_0 (z - z_1)^{\sigma_1} \dots (z - z_r)^{\sigma_r} \ .$$

The roots can be complex numbers.

$p(z)$ may be called the *characteristic polynomial*, $z_1, \dots, z_r$ are its distinct *roots*, and σ_i is the *multiplicity* of the root z_i.

We claimed that for each $1 \leq i \leq r$, for each $0 \leq l \leq \sigma_i - 1$

$$y[n] = A \frac{n!}{l!(n-l)!} z_i^{n-l}$$

is a solution to the homogeneous equation. Here A is an arbitrary (complex) constant.

We verify this for $l = 0$ for every root z_i. We have to show that $y[n] = z_i^n$ is a solution to the homogeneous equation.

$$\begin{aligned} \sum_{k=0}^{N} a_k z_i^{n-k} &= z^{n-N} \sum_{k=0}^{N} a_k z_i^{N-k} \\ &= z^{n-N} p(z_i) \\ &= 0 \ . \end{aligned}$$

We verify this for $l = 1$ for every root z_i that has multiplicity $\sigma_i \geq 2$. We have to show that $y[n] = n z_i^{n-1}$ is a solution to the homogeneous equation.

First write

$$p(z) = (z - z_i)^2 q(z)$$

This is possible because $\sigma_i \geq 2$.

Differentiating with respect to z on both sides, we get

$$\frac{d}{dz}p(z) = 2(z - z_i)q(z) + (z - z_i)^2 \frac{d}{dz}q(z) \ .$$

From this we conclude that

$$\frac{d}{dz}p(z) \mid_{z=z_i} = 0 \ .$$

Of course, we also have

$$p(z) \mid_{z=z_i} = p(z_i) = 0 \ .$$

We now write :

$$\begin{aligned}
\sum_{k=0}^{N} a_k(n-k)z_i^{n-k-1} &= \frac{d}{dz}\left(\sum_{k=0}^{N} a_k z^{n-k}\right) \mid_{z=z_i} \\
&= \frac{d}{dz}\left(z^{n-N}\sum_{k=0}^{N} a_k z^{N-k}\right) \mid_{z=z_i} \\
&= \frac{d}{dz}\left(z^{n-N}p(z)\right) \mid_{z=z_i} \\
&= z^{n-N}\frac{d}{dz}p(z) \mid_{z=z_i} + (n-N)z^{n-N-1}p(z) \mid_{z=z_i} \\
&= 0
\end{aligned}$$

In general, suppose z_i is a root of $p(z)$ with multiplicity $\sigma_i \geq 2$, and let $1 \leq l \leq \sigma_i - 1$ (we have already handled the case $l = 0$).

Write

$$p(z) = (z - z_i)^{l+1}q(z)$$

This is possible because $\sigma_i \geq l + 1$.

Differentiating with respect to z on both sides, we see that

$$\frac{d^j}{dz^j}p(z) \mid_{z=z_i} = 0 \ ,$$

for all $j = 0, 1, \ldots, l$.

We now write :

$$\begin{aligned}
\sum_{k=0}^{N} a_k \frac{1}{l!}(n-k)(n-k-1)\dots(n-k-l+1)z_i^{n-k-l} &= \frac{1}{l!}\frac{d^l}{dz^l}\left(\sum_{k=0}^{N} a_k z^{n-k}\right)|_{z=z_i} \\
&= \frac{1}{l!}\frac{d^l}{dz^l}\left(z^{n-N}\sum_{k=0}^{N} a_k z^{N-k}\right)|_{z=z_i} \\
&= \frac{1}{l!}\frac{d^l}{dz^l}\left(z^{n-N}p(z)\right)|_{z=z_i} \\
&= 0
\end{aligned}$$