Consider the constant coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k]$$

The associated homogeneous equation is:

$$\sum_{k=0}^{N} a_k y[n - k] = 0.$$ 

Consider the polynomial $p(z) = \sum_{k=0}^{N} a_k z^{N-k}$.

Factor it as

$$p(z) = a_0(z - z_1)^{\sigma_1} \ldots (z - z_r)^{\sigma_r}.$$ 

The roots can be complex numbers.

$p(z)$ may be called the characteristic polynomial, $z_1, \ldots, z_r$ are its distinct roots, and $\sigma_i$ is the multiplicity of the root $z_i$.

We claimed that for each $1 \leq i \leq r$, for each $0 \leq l \leq \sigma_i - 1$

$$y[n] = A \frac{n!}{l!(n-l)!} z_i^{n-l}$$

is a solution to the homogeneous equation. Here $A$ is an arbitrary (complex) constant.

We verify this for $l = 0$ for every root $z_i$. We have to show that $y[n] = z_i^n$ is a solution to the homogeneous equation.

$$\sum_{k=0}^{N} a_k z_i^{n-k} = z_i^{n-N} \sum_{k=0}^{N} a_k z_i^{N-k}$$

$$= z_i^{n-N} p(z_i)$$

$$= 0.$$ 

We verify this for $l = 1$ for every root $z_i$ that has multiplicity $\sigma_i \geq 2$. We have to show that $y[n] = n z_i^{n-1}$ is a solution to the homogeneous equation.

First write

$$p(z) = (z - z_i)^2 q(z)$$

This is possible because $\sigma_i \geq 2$. 

Differentiating with respect to $z$ on both sides, we get
\[
\frac{d}{dz}p(z) = 2(z - z_i)q(z) + (z - z_i)^2 \frac{d}{dz}q(z).
\]
From this we conclude that
\[
\frac{d}{dz}p(z) \bigg|_{z=z_i} = 0.
\]
Of course, we also have
\[
p(z) \bigg|_{z=z_i} = p(z_i) = 0.
\]
We now write:
\[
\sum_{k=0}^{N} a_k (n-k) z_i^{n-k-1} = \frac{d}{dz} \left( \sum_{k=0}^{N} a_k z_i^{n-k} \right) \bigg|_{z=z_i} = \frac{d}{dz} \left( z_i^{n-N} \sum_{k=0}^{N} a_k z_i^{N-k} \right) \bigg|_{z=z_i} = \frac{d}{dz} \left( z_i^{n-N} p(z) \right) \bigg|_{z=z_i} = z_i^{n-N} \frac{d}{dz} p(z) \bigg|_{z=z_i} + (n-N) z_i^{n-N-1} p(z) \bigg|_{z=z_i} = 0
\]
In general, suppose $z_i$ is a root of $p(z)$ with multiplicity $\sigma_i \geq 2$, and let $1 \leq l \leq \sigma_i - 1$ (we have already handled the case $l = 0$).

Write
\[
p(z) = (z - z_i)^{l+1}q(z)
\]
This is possible because $\sigma_i \geq l + 1$.

Differentiating with respect to $z$ on both sides, we see that
\[
\frac{d^j}{dz^j} p(z) \bigg|_{z=z_i} = 0,
\]
for all $j = 0, 1, \ldots, l$. 
We now write:

\[ \sum_{k=0}^{N} a_k \frac{1}{l!} (n-k)(n-k-1) \ldots (n-k-l+1) z_i^{n-k-l} = \frac{1}{l!} \frac{d^l}{dz^l} \left( \sum_{k=0}^{N} a_k z^{n-k} \right) |_{z=z_i} \]

\[ = \frac{1}{l!} \frac{d^l}{dz^l} \left( z^{n-N} \sum_{k=0}^{N} a_k z^{N-k} \right) |_{z=z_i} \]

\[ = \frac{1}{l!} \frac{d^l}{dz^l} \left( z^{n-N} p(z) \right) |_{z=z_i} \]

\[ = 0 \]