Lecture 6 notes

Square matrices

A square $d \times d$ matrix (real or complex) maps $\mathbb{C}^d$ to $\mathbb{C}^d$.

$$
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1d} \\
  a_{21} & a_{22} & \ldots & a_{2d} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{d1} & a_{d2} & \ldots & a_{dd}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_d
\end{bmatrix}
= 
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_d
\end{bmatrix}
$$

where

$$
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_d
\end{bmatrix}
= 
\begin{bmatrix}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1d}x_d \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2d}x_d \\
  \vdots \\
  a_{d1}x_1 + a_{d2}x_2 + \ldots + a_{dd}x_d
\end{bmatrix}
$$

The matrix $A$ is linear, i.e

$$
Ax_{(1)} = y_{(1)} \text{ and } Ax_{(2)} = y_{(2)} \implies A(ax_{(1)} + bx_{(2)}) = ay_{(1)} + by_{(2)}
$$

for all $a, b \in \mathbb{C}$.

Scalars ($1 \times 1$ square matrices) commute

$d \times d$ matrices in general do not commute.

$$
\begin{bmatrix}
  2 & 1 \\
  0 & 2
\end{bmatrix}
\begin{bmatrix}
  2 & 0 \\
  1 & 2
\end{bmatrix}
= 
\begin{bmatrix}
  5 & 2 \\
  2 & 4
\end{bmatrix}
$$

while

$$
\begin{bmatrix}
  2 & 0 \\
  1 & 2
\end{bmatrix}
\begin{bmatrix}
  2 & 1 \\
  0 & 2
\end{bmatrix}
= 
\begin{bmatrix}
  4 & 2 \\
  2 & 5
\end{bmatrix}
$$

Eigenvalues and Eigenvectors

Let $A$ be a $d \times d$ matrix (real or complex).

$det(sI - A)$ is called the characteristic polynomial of $A$.

Any root of the characteristic polynomial is called an eigenvalue of $A$. 


Every square matrix has at least one eigenvalue, possibly complex (from the fundamental theorem of algebra).

Let $\lambda$ be an eigenvalue of $A$.

Then $\lambda I - A$ is singular, so there is at least one nonzero vector $v \in \mathbb{C}^d$ with $$ (\lambda I - A)v = 0 $$

Such a vector is called an eigenvector of $A$ corresponding to the eigenvalue $\lambda$.

Note that $$ Av = \lambda v $$

A basis of vectors
A set of vectors $v_1, \ldots, v_k \in \mathbb{C}^d$ is called linearly independent if
$$ a_1 v_1 + \ldots + a_k v_k = 0 \implies a_1 = \ldots = a_k = 0 $$

If a set of vectors $v_1, \ldots, v_k \in \mathbb{C}^d$ is linearly independent then $k \leq d$.

A set of vectors $v_1, \ldots, v_k \in \mathbb{C}^d$ is said to span $\mathbb{C}^d$ if every vector $v \in \mathbb{C}^d$ can be written as
$$ v = a_1 v_1 + \ldots + a_k v_k $$

for some coefficients $a_1, \ldots, a_k \in \mathbb{C}$.

If a set of vectors $v_1, \ldots, v_k \in \mathbb{C}^d$ spans $\mathbb{C}^d$ then $k \geq d$.

A basis for $\mathbb{C}^d$ is a set of vectors $v_1, \ldots, v_d$ that is both linearly independent and spans $\mathbb{C}^d$.

A point to remember
Let $A$ and $B$ be square $d \times d$ matrices.

Suppose there is a basis $v_1, \ldots, v_d$ for $\mathbb{C}^d$ comprised of eigenvectors of $A$, with respective eigenvalues $\lambda_1, \ldots, \lambda_d$. 
Suppose the same \( v_1, \ldots, v_d \) for \( \mathbb{C}^d \) are also eigenvectors of \( B \) with respective eigenvalues \( \mu_1, \ldots, \mu_d \).

Then \( A \) and \( B \) commute.

Given \( v \in \mathbb{C}^d \) write \( v = a_1 v_1 + \ldots + a_k v_k \) for some coefficients \( a_1, \ldots, a_k \in \mathbb{C} \).

\[
ABv = AB \left( \sum_{i=1}^{k} a_i v_i \right) = A \left( \sum_{i=1}^{k} a_i \mu_i v_i \right) = \sum_{i=1}^{k} a_i \mu_i \lambda_i v_i
\]

\[
BAv = BA \left( \sum_{i=1}^{k} a_i v_i \right) = B \left( \sum_{i=1}^{k} a_i \lambda_i v_i \right) = \sum_{i=1}^{k} a_i \mu_i \lambda_i v_i
\]

**Orthonormal basis**

- Let \( v_1, \ldots, v_d \) be a basis for \( \mathbb{C}^d \).
- The basis is called orthonormal if:

\[
\sum_{i=1}^{d} v_{li} v_{mi}^* = \begin{cases} 1 & \text{if } l = m \\ 0 & \text{if } l \neq m \end{cases}
\]

where \( 1 \leq l, m \leq d \).

We sometimes write this as

\[
<v_l, v_m> = \delta_{lm}
\]

for all \( 1 \leq l, m \leq d \).
Let \( v_1, \ldots, v_d \) be an orthonormal basis for \( \mathbb{C}^d \) and let \( v \) be an arbitrary vector in \( \mathbb{C}^d \).

We can write
\[
v = \sum_{i=1}^{d} a_i v_i
\]
where
\[
a_i = \langle v, v_i \rangle
\]
because, for any \( 1 \leq j \leq d \), we have:
\[
\langle v, v_j \rangle = \sum_{i=1}^{d} a_i \langle v_i, v_j \rangle = \sum_{i=1}^{d} a_i \delta_{ij} = a_j
\]

*Eigenfunctions of a discrete time LTI system*

\[
y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]
\]

Consider the complex exponential input
\[
x[n] = A z^n
\]
The associated output:
\[
y[n] = \sum_{k=-\infty}^{\infty} h[k] A z^{n-k} = A z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} = H(z) A z^n
\]
is a fixed constant multiple of the input.
The signal $Az^n$ is an eigenfunction of the system corresponding to the eigenvalue $H(z)$.

$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$ is called the transfer function of the LTI system.

Consider the input

$$x[n] = \sum_r A_r z^n$$

The corresponding output is

$$y[n] = \sum_r A_r H(z_r) z^n$$

If we could express an arbitrary input as a linear combination of complex exponentials, we would be able to figure out the corresponding output from knowledge of the transfer function $H(z)$.

Eigenfunctions of a continuous time LTI system

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

Consider the complex exponential input

$$x(t) = Ae^{st}$$

The associated output:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) Ae^{s(t-\tau)} d\tau$$

$$= Ae^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$= H(s) Ae^{st}$$

is a fixed constant multiple of the input.

The signal $Ae^{st}$ is an eigenfunction of the system corresponding to the eigenvalue $H(s)$.

$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$ is called the transfer function of the LTI system.

Consider the input

$$x(t) = \sum_r A_r e^{s_r t}$$

The corresponding output is

$$y(t) = \sum_r A_r H(s_r) e^{s_r t}$$
If we could express an arbitrary input as a linear combination of complex exponentials, we would be able to figure out the corresponding output from knowledge of the transfer function $H(s)$.

*Fourier series expansion for a continuous time periodic signal*

Fix $T > 0$. Let $\omega_0$ denote $\frac{2\pi}{T}$.

The complex exponentials that are periodic with period $T$ are the signals $e^{j k \omega_0 t}$ for $k \in \mathbb{Z}$.

Given a periodic signal $x(t)$ with period $T$ we might hope to be able to write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}$$

where

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-j k \omega_0 t} dt$$

If a periodic signal $x(t)$ with period $T$ satisfies

$$\int_0^T |x(t)|^2 dt < \infty$$

and we define its *Fourier series coefficients*

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-j k \omega_0 t} dt$$

then we have

$$\lim_{K \to \infty} \int_0^T |x(t) - \sum_{k=-K}^{K} a_k e^{j k \omega_0 t}|^2 dt = 0.$$ 

*Fourier Series examples*

- $x(t) = \sin(\omega_0 t)$, where $\omega_0 = \frac{2\pi}{T}$.
  has
  $a_{-1} = a_1 = \frac{1}{T}$ and $a_k = 0$ for $k \neq \pm 1$.
- $x(t)$ periodic with period $T$ and
  $$x(t) = \begin{cases} 1 & \text{if } |t| < T_1 \\ 0 & \text{if } T_1 < |t| < \frac{T}{2} \end{cases}$$
  has
\[ a_0 = \frac{2T_1}{T} \quad \text{and} \quad a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad \text{for} \quad k \neq 0 \]

The special case of the above when \( T_1 = \frac{T}{4} \) has

\[ a_0 = \frac{1}{2} \quad \text{and} \quad a_k = 0 \quad \text{for} \quad k \text{ even} \quad \text{and} \quad a_k = \frac{1}{|k|\pi} \quad \text{for} \quad k \text{ odd}. \]