Problem 1 (Wireless Downlink.)

A base station transmits simultaneously to three mobiles. It needs to send a signal \( s_1(t) \) to Mobile 1, a signal \( s_2(t) \) to Mobile 2, and a signal \( s_3(t) \) to Mobile 3. Since \( s_1(t), s_2(t) \) and \( s_3(t) \) are speech signals, they are real-valued and band-limited:

\[
S_i(j\omega) = 0, \quad \text{for } |\omega| > W, \quad (1)
\]

for \( i = 1, 2, 3 \). The base station needs to produce a real-valued output signal \( y(t) \) to be transmitted out of the antenna. The FCC allows you to use the frequency band \( \omega_0 \leq |\omega| \leq \omega_0 + 3W \).

- Draw the block diagram of the base station, with inputs \( s_1(t), s_2(t) \) and \( s_3(t) \) and output \( y(t) \), where \( y(t) \) must comply with FCC regulations and permit perfect recovery of \( s_1(t), s_2(t) \) and \( s_3(t) \). Hint: There are multiple solutions; only one is required.

- Draw the block diagram of the demodulation system at Mobile 1, with input \( y(t) \) and output \( s_1(t) \).

You may use arbitrary components, but carefully specify all involved parameters, such as cut-off frequencies of filters.

Problem 2 (Amplitude modulation.)

(a) For the discrete-time signal \( x[n] \) it is known that \( X(e^{j\omega}) = 0, \) for \( |\omega| > \pi/4 \). Determine the range of \( \omega \) for which the DTFT of \( y[n] = \cos(\frac{5\pi}{4}n)x[n] \) must be zero. Hint: Select an example spectrum \( X(e^{j\omega}) \) and sketch the resulting DTFT of \( y[n] \).

(b) The real-valued data signal \( x(t) \) is known to be band-limited, i.e., \( X(j\omega) = 0, \) for \( |\omega| > W \). Consider the block diagram of Figure 1, where

\[
H_1(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \omega_c, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad H_2(j\omega) = \begin{cases} 1, & \text{for } |\omega| \geq 2\omega_c, \\ 0, & \text{otherwise}. \end{cases} \quad (2)
\]

- Pick an arbitrary (bandlimited) example spectrum for \( x(t) \), and sketch the corresponding spectrum of the signal \( y(t) \).

- For what values of the parameters \( W \) and \( \omega_c \) is it possible to recover \( x(t) \) from \( y(t) \)?

- Provide the block diagram of a system that recovers \( x(t) \), given \( y(t) \), carefully specifying all involved parameters.

(c) The real-valued data signal \( x(t) \) is known to be band-limited, i.e., \( X(j\omega) = 0, \) for \( |\omega| > W \). The goal is to perform standard (i.e., double-sideband) AM with carrier frequency \( \omega_c > 5W \). Unfortunately, the only type of modulator available is multiplication by \( \cos(\frac{5\pi}{4}t) \). Otherwise, addition, scalar multiplication, and filters can be used. Draw the block diagram of the system that achieves our goal, and if your system uses a filter, specify the desired frequency response. Hint: Pick an example spectrum for \( x(t) \) and sketch the spectra of intermediate signals to maximize your chances for partial credit.
The real-valued data signal \( x(t) \) is known to be band-limited, i.e., \( X(j\omega) = 0 \), for \( |\omega| > W \). The goal is to perform single-sideband AM with only the lower sideband, with carrier frequency \( \omega_c > 5W \). Again, you can use addition, scalar multiplication, and multiplication by \( \cos(\omega_m t) \), for arbitrary \( \omega_m \). However, this time, you only have fixed ideal low-pass filters with the following frequency response:

\[
H(j\omega) = \begin{cases} 
1, & \text{for } |\omega| \leq \omega_c/2 \\
0, & \text{otherwise}. 
\end{cases}
\]  

(3)

Draw the block diagram of a system that achieves the goal, clearly specifying all involved parameters, such as the frequencies of the modulators, etc. Hint: Pick an example spectrum for \( x(t) \) and sketch the spectra of intermediate signals to maximize your chances for partial credit.

Problem 3 \((PAM.)\)

Two pulses are suggested for a PAM system:

\[
p_1(t) = ae^{-t}u(t), \quad \text{and} \quad p_2(t) = be^{-10t}u(t),
\]

(4)

where \( a \) and \( b \) are positive real numbers that will be selected appropriately, leading to

\[
y_i(t) = \sum_{k=-\infty}^{\infty} x[k]p_i(t - kT), \text{ for } i = 1, 2,
\]

(5)

where we choose \( T = 1 \). We suppose that the data signal is bounded to \( |x[n]| \leq 1 \). In this problem, we want to compare the two pulses \( p_1(t) \) and \( p_2(t) \).

(a) Select \( a = 2 \) and \( b = 2\sqrt{10} \). For this choice, it can be shown that the pulse energy is the same for \( p_1(t) \) and for \( p_2(t) \). (You don’t have to show this!) Now consider the transmission of \( p_1(t) \) and \( p_2(t) \), respectively, across a communication channel with impulse response \( h(t) \) and corresponding frequency response

\[
H(j\omega) = \frac{1}{6 + j\omega}.
\]

(6)

This yields an output signal \( z_i(t) = (p_i * h)(t) \), for \( i = 1, 2 \).

- Evaluate the energy of the received signals, \( z_1(t) \) and \( z_2(t) \), respectively.
- Which received signal has the larger energy?
- How is it possible that even though the two pulses have the same transmitted energy, their received energies differ?
(b) (Hard problem) To have a fair comparison, we have to make sure that the powers of the transmitted signals \( y_1(t) \) and \( y_2(t) \), respectively, are equal. To adjust the power, assume that \( x[n] = 1 \) for all \( n \), i.e., for \(-\infty < n < \infty\). Determine the relationship between \( a \) and \( b \) such that for this particular \( x[n] \), the signals \( y_1(t) \) and \( y_2(t) \) have the same power. (As seen in class, this provides a worst case analysis.)

Hint: By contrast to Part (a), this question studies the power of the entire signal, rather than the energy of a single pulse.

Problem 4 (Discrete-time Differentiator.)

We would like to construct a system \( D \) that implements a derivative, that is, for an input \( x(t) \), the system should give an output \( y(t) \) given by

\[
y(t) = D\{x(t)\} = \frac{dx(t)}{dt}.
\]

It is suggested to use the following system:

\[
\begin{array}{cccccc}
  & x(t) & \rightarrow & H(j\omega) & \rightarrow & z(t) \\
  & & & \text{Sampling at} & \text{intervals} & T \\
  & & & z[n] & \rightarrow & G(e^{j\Omega}) & y_1[n] \\
  & & & & \rightarrow & \text{Ideal} & \text{Reconstruction} \\
  & & & & & \rightarrow & \tilde{y}(t)
\end{array}
\]

where

\[
G(e^{j\Omega}) = \frac{\Omega}{T}, \text{ for } |\Omega| \leq \pi.
\]

This system does not exactly implement the desired system \( D \). Instead, it produces an output \( \tilde{y}(t) \) which is, in general, not equal to the desired output \( y(t) \).

(a) Suppose that

\[
H(j\omega) = \begin{cases} 
1, & \text{for } |\omega| \leq \pi/T \\
0, & \text{otherwise}
\end{cases}
\]

• Determine and sketch the overall frequency response (magnitude and phase) of the system with input \( x(t) \) and output \( \tilde{y}(t) \).

• For the test signal \( x(t) \) with Fourier transform \( X(j\omega) = e^{-|\omega|} \), determine the error between the desired signal, \( y(t) \), and the actual system output, \( \tilde{y}(t) \), given by

\[
E = \int_{-\infty}^{\infty} |y(t) - \tilde{y}(t)|^2 dt,
\]

as a function of the sampling interval \( T \). What happens as we increase the sampling frequency?

(b) Unfortunately, it is quite difficult to exactly implement ideal frequency filters like \( H(j\omega) \) in Part (a). As a simple model of this imperfection, suppose now that

\[
H(j\omega) = \begin{cases} 
1, & \text{for } |\omega| \leq \pi/T \\
\epsilon, & \text{otherwise}
\end{cases}
\]

For the same test signal as in Part (a), that is, \( X(j\omega) = e^{-|\omega|} \), determine the spectrum \( Z_\delta(j\omega) \) of the sampled signal,

\[
z_\delta(t) = z(t) \sum_{k=-\infty}^{\infty} \delta(t - kT).
\]
• Start with a sketch of $Z_\delta(j\omega)$, carefully labeling the frequency axis.
• Which adverse effect corrupts the signal $z_\delta(t)$?
• Then, write out a formula for $Z_\delta(j\omega)$. The simpler your formula, the better.

**Problem 5 (LTI System Analysis.)**

A causal LTI system has a transfer function

$$H(s) = \frac{(s + 4)(s^2 + 5s + 6)}{(s + 1)(s^2 - 2s + 3)}.$$  \hspace{1cm} (14)

Determine the differential equation that describes this system. Find the impulse response $h(t)$. Is the system stable? Does this system have a stable and causal inverse system?