Linear Time-Invariant (LTI) Systems

$$x(t) \rightarrow y(t)$$

Linearity: Two conditions must be satisfied:

1. **Scaling:**
   $$ax(t) \rightarrow ay(t)$$ for any number $$a;$$ (1)

2. **Superposition:**
   $$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t).$$ (2)

Corollary: If the input to a linear system is 0, the output must be 0.

**Proof.** Choose $$a = 0$$ in the scaling property.

**Time-Invariance:** A time shift in the input results is an identical time shift in the output:

$$x(t - T) \rightarrow y(t - T).$$ (3)

**Example:** Moving average filter:

$$y[n] = \frac{1}{3} (x[n-1] + x[n] + x[n+1]) \rightarrow \text{LTI.}$$ (4)

**Example:** Median Filter:

$$y[n] = \text{med}\{x[n-1], x[n], x[n+1]\} \rightarrow \text{TI, but nonlinear.}$$ (5)
Discrete-Time (DT) LTI Systems: Convolution Sum

Let \( h[n] \) denote the response of an LTI system to the unit impulse:

Then, for any input \( x[n] \), the output is:

\[
y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{“convolution sum”} \tag{6}
\]

Proof. Rewrite \( x[n] \) as

\[
x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots \tag{7}
\]

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{8}
\]

Since \( \delta[n] \to h[n] \), by time-invariance: \( \delta[n-k] \to h[n-k] \).

Then, by linearity: \( \sum_k x[k]\delta[n-k] \to \sum_k x[k]h[n-k] \).

Example: For the moving average system above,
\( y[n] = \sum_{k=\infty}^{\infty} x[k] h[n-k] \)

\[ = \sum_{k=n-1}^{n+1} \frac{1}{3} x[k] = \frac{1}{3} (x[n-1] + x[n] + x[n+1]) \quad (9) \]

Example: For the median filter:

Since the system is nonlinear, we can’t use convolution to predict the output.

**Continuous-Time (CT) LTI Systems: Convolution Integral**

Unit impulse:

\[ \delta(t) \triangleq \lim_{\Delta \to 0} \delta_{\Delta}(t) \quad (10) \]

where \( \delta_{\Delta}(t) \) is as in Figure 1.

Let \( h(t) \) denote the response of a LTI system to \( \delta(t) \).

Then, for any input \( x(t) \), the output is:

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau \quad \text{"convolution integral"} \quad (11) \]

**Proof.** First, note that the staircase approximation in Figure 2 recovers \( x(t) \) as \( \Delta \to 0 \):

\[ x(t) = \lim_{\Delta \to 0} \sum_{k=\infty}^{\infty} x(k\Delta) \Delta \delta_{\Delta}(t-k\Delta). \quad (12) \]

Next, let \( h_{\Delta}(t) \) denote the response of the system to \( \delta_{\Delta}(t) \) and note from the LTI property that the response to each term in the sum above is \( x(k\Delta) \Delta h_{\Delta}(t-k\Delta) \). Thus, the response to \( x(t) \) is

\[ y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) h_{\Delta}(t-k\Delta) \Delta = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau. \quad (13) \]
Properties of LTI Systems

We will denote the convolution operation by "\(\ast\)."

1. Commutative Property:

\[
\text{x[n] \ast h[n]} = \text{h[n] \ast x[n]} \tag{14}
\]

Proof.

\[
\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r], \tag{15}
\]

with the change of variables \((n-k) \triangleq r\).

2. Distributive Property:

\[
\text{x[n] \ast (h_1[n] + h_2[n]) = x[n] \ast h_1[n] + x[n] \ast h_2[n]} \tag{16}
\]

3. Associative Property:

\[
\text{x[n] \ast (h_1[n] \ast h_2[n]) = (x[n] \ast h_1[n]) \ast h_2[n]} \tag{17}
\]

Properties 1, 2, 3 above also hold for CT systems.
Determining Causality from the Impulse Response

For a DT LTI system, causality means:
\[ h[n] = 0, \quad \forall n < 0. \]  
(18)

For a CT LTI system, causality means:
\[ h(t) = 0, \quad \forall t < 0. \]  
(19)

**Proof.** Since \( y[n] = \sum_{k=\infty}^{\infty} h[k] x[n-k] \), if \( h[k] \neq 0 \) for some \( k < 0 \), then \( y[n] \) depends on \( x[n-k] \), where \( n-k > n \).

**Example:** Moving average system above: \( h[-1] \neq 0 \) \( \rightarrow \) noncausal.

Determining Stability from the Impulse Response

Stability criterion for a DT LTI system:
\[ \sum_{k=-\infty}^{\infty} |h[k]| < \infty. \]  
(20)

Stability criterion for a CT LTI system:
\[ \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty. \]  
(21)

**Proof.**

**Sufficiency:** Suppose \( \sum_{k=-\infty}^{\infty} |h[k]| < \infty \) and show that bounded inputs give bounded outputs:

- \( |x[n]| \leq B \) for all \( n \), for some \( B > 0 \).
- \( |y[n]| = |\sum_k x[n-k] h[k]| \leq \sum_k |x[n-k]| \cdot |h[k]| \leq B \sum_k |h[k]| < \infty \).

**Necessity:** To prove “stable \( \Rightarrow \sum_k |h[k]| < \infty \)” prove the contrapositive:

\[ \sum_{k} |h[k]| = \infty \Rightarrow \text{unstable}. \]
(22)

Let \( x[n] = \text{sgn}\{h[-n]\} \). Then, since \( y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] \):
\[ y[0] = \sum_{k=-\infty}^{\infty} h[k] x[-k] = \sum_{k} h[k] \text{sign}\{h[k]\} = \sum_{k} |h[k]| = \infty. \]
(23)

**Examples:**

1. Moving average system above:
\[ \sum_{k} |h[k]| = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \rightarrow \text{stable}. \]
(24)

2. Integrator: \( y(t) = \int_{-\infty}^{t} x(\tau) \tau \), \( h(t) \) is the unit step (see Figure 3), and
\[ \int_{-\infty}^{\infty} |h(\tau)| d\tau = \infty. \]

**Figure 3: UnitStep**