All-Pass Systems

What is the frequency response of an LTI system with transfer function:

\[ H(s) = \frac{a - s}{s + a}, \quad a > 0? \]

General all-pass system:

\[ H_{ap}(s) = \prod_{i=1}^{n} \left( \frac{a_i - s}{s + a_i} \right), \quad a_i > 0 \quad i = 1, ..., n. \]

For a stable and causal all-pass system, all zeros are in the right half-plane because they are mirror images of the poles.

Although \(|H_{ap}(j\omega)| \equiv 1\), an all-pass system introduces delay:

\[ e^{j\omega t} \rightarrow H_{ap}(j\omega) e^{j\omega t} = e^{j\angle H_{ap}(j\omega)} e^{j\omega t} = e^{j\omega (t - \tau(\omega))} \]

where:

\[ \tau(\omega) \triangleq -\frac{\angle H_{ap}(j\omega)}{\omega} > 0. \]

Moreover, the system is not linear phase (i.e. \(\tau(\omega)\) is not constant); therefore it causes phase distortion. (Recall Lecture 8.)

Minimum Phase Systems

A stable and causal LTI system is called minimum phase if all of its zeros are in the open left half-plane.
Any non-minimum phase transfer function $H(s)$ can be decomposed as:

$$H(s) = H_{\text{min}}(s) \cdot H_{\text{ap}}(s)$$

(2)

This decomposition explains the genesis of the term *minimum phase*:

$$|H(j\omega)| = |H_{\text{min}}(j\omega)|$$

since

$$|H_{\text{ap}}(j\omega)| \equiv 1,$$

but the all-pass component adds more delay. Therefore, $H(s)$ and $H_{\text{min}}(s)$ have identical frequency responses in magnitude, but $H_{\text{min}}(s)$ has the minimum phase delay.

**Example:**

$$H(s) = \frac{10 - s}{10(s + 1)} = \frac{s + 10}{10(s + 1)} \cdot \frac{10 - s}{s + 10}$$

If $H_d(s)$ is minimum phase, we can simply choose $H_c(s) = \frac{1}{H_d(s)}$.

If $H_d(s)$ is nonminimum phase, $H_c(s) = \frac{1}{H_d(s)}$ is unstable. To avoid instability, decompose: $H_d(s) = H_{d,\text{min}}(s)H_{d,\text{ap}}(s)$ and select:

$$H_c(s) = \frac{1}{H_{d,\text{min}}(s)} \quad \Rightarrow \quad H_d(s)H_c(s) = H_{d,\text{ap}}(s)$$

magnitude distortion eliminated
Transfer Functions of Interconnected LTI Systems

Section 9.8 in Oppenheim & Willsky

\[ h(t) = h_1(t) + h_2(t) \]
\[ H(s) = H_1(s) + H_2(s) \]

\[ h(t) = h_1(t) \ast h_2(t) \]
\[ H(s) = H_1(s)H_2(s) \]

\[ X(s) = H_1(s)E(s) \]
\[ Y(s) = H_1(s)X(s) - H_1(s)H_2(s)Y(s) \]
\[ (1 + H_1(s)H_2(s))Y(s) = H_1(s)X(s) \]
\[ Y(s) = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \]

Example: Feedback Control

\[ r(t): \text{reference signal to be tracked by } y(t) \]
\[ H_c(s): \text{controller, } H_p(s): \text{system to be controlled - “plant”} \]
\[ H(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} \]

Example:

\[ M \frac{dy}{dt} = x(t) \quad \rightarrow \quad MsY(s) = X(s) \]
\[ H_p(s) = \frac{1}{Ms} \]

Take constant gain controller: \( H_c(s) = K \)

\[ H(s) = \frac{K}{1 + \frac{K}{Ms}} = \frac{1}{\tau s + 1} \quad \tau = \frac{M}{K} \]
The Unilateral Laplace Transform

\[ \mathcal{X}(s) = \int_{0}^{\infty} x(t) e^{-st} dt \]  

(3)

Identical to the bilateral Laplace transform if \( x(t) = 0 \) for \( t < 0 \).

Example: \( x(t) = e^{-a(t+1)}u(t+1) \)

\[ X(s) = \frac{e^s}{s + a} \quad \text{Re}\{s\} > -a \]
\[ \mathcal{X}(s) = \frac{e^{-a}}{s + a} \quad \text{Re}\{s\} > -a \]

Properties of the unilateral Laplace transform

Most properties of the bilateral Laplace transform also hold for the unilateral Laplace transform.

Exceptions:

Convolution:

\( x_1(t) \ast x_2(t) \leftrightarrow X_1(s)X_2(s) \) if \( x_1(t) = x_2(t) = 0 \) for all \( t < 0 \)

This follows from the convolution property of the bilateral Laplace transform which coincides with the unilateral transform because \( x_1(t) = x_2(t) = 0, \ t < 0 \).

Differentiation in Time:

\[ \frac{dx(t)}{dt} \leftrightarrow s\mathcal{X}(s) - x(0^-) \]

Repeated application gives:

\[ \frac{d^2x(t)}{dt^2} = \frac{d}{dt} \left\{ \frac{dx(t)}{dt} \right\} \leftrightarrow s \left( s\mathcal{X}(s) - x(0^-) \right) - \frac{dx}{dt}(0^-) \]

\[ = s^2\mathcal{X}(s) - sx(0^-) - \frac{dx}{dt}(0^-) \]

\[ \frac{d^3x(t)}{dt^3} = \frac{d}{dt} \left\{ \frac{d^2x(t)}{dt^2} \right\} \leftrightarrow s \left( s^2\mathcal{X}(s) - sx(0^-) - \frac{dx}{dt}(0^-) \right) - \frac{d^2x}{dt^2}(0^-) \]

\[ = s^3\mathcal{X}(s) - s^2x(0^-) - s\frac{dx}{dt}(0^-) - \frac{d^2x}{dt^2}(0^-) \]
Solving differential equations with the unilateral Laplace transform

Example:

\[
d^2y(t) \over dt^2 + 3 \frac{dy}{dt} + 2y(t) = e^t \quad t \geq 0
\]  

(4)

Initial condition \( y(0^-) = a, \frac{dy}{dt}(0^-) = b \).

\[
(s^2Y(s) - as - b) + 3(sY(s) - a) + 2Y(s) = \frac{1}{s-1}
\]

\[
(s^2 + 3s + 2)Y(s) = as + b + 3a + \frac{1}{s-1} = \frac{as^2 + (b + 2a)s + (1 - b - 3a)}{(s-1)^2}
\]

\[
Y(s) = \frac{as^2 + (b + 2a)s + (1 - b - 3a)}{(s+1)(s+2)(s-1)}
\]

Partial fraction expansion:

\[
Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{B}{s-1}
\]

\[
= \frac{(A_1 + A_2 + B)s^2 + (A_1 + 3B)s + (2B - 2A_1 - A_2)}{(s+1)(s+2)(s-1)}
\]

Match coefficients:

\[
\begin{align*}
A_1 + A_2 + B &= a \\
A_1 + 3B &= b + 2a \\
2B - 2A_1 - A_2 &= 1 - b - 3b
\end{align*}
\]

\[
\begin{align*}
B &= 1/6 \\
A_1 &= -1/2 + 2a + b \\
A_2 &= 1/3 - a - b
\end{align*}
\]

Then,

\[
y(t) = \frac{1}{6}e^t + \left(-\frac{1}{2} + 2a + b\right)e^{-t} + \left(\frac{1}{3} - a - b\right)e^{-2t} \quad t \geq 0.
\]

Compare this to the standard method for solving linear constant coefficient differential equations:

The first term in \( y(t) \) above is the particular solution. If we substitute \( y_p(t) = \frac{1}{6}e^t \) in (4):

\[
\frac{d^2y_p(t)}{dt^2} + 3 \frac{dy_p}{dt} + 2y_p(t) = e^t.
\]

The second and third terms constitute the homogeneous solution. If we substitute \( y_h(t) = A_1e^{-t} + A_2e^{-2t} \):

\[
\frac{d^2y_h(t)}{dt^2} + 3 \frac{dy_h}{dt} + 2y_h(t) = 0.
\]

Thus, \( y(t) = y_p(t) + y_h(t) \) and \( A_1 \) and \( A_2 \) are selected to satisfy the initial conditions.