Feedback Control

\[ r(t) \rightarrow e(t) \rightarrow H_c(s) \rightarrow x(t) \rightarrow H_p(s) \rightarrow y(t) \]

\( r(t) \): reference signal to be tracked by \( y(t) \)

\( H_c(s) \): controller; \( H_p(s) \): system to be controlled ("plant")

Closed-loop transfer function:

\[
H(s) = \frac{Y(s)}{R(s)} = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}
\]

Constant-gain control: \( H_c(s) = K \)

\[
H(s) = \frac{KH_p(s)}{1 + KH_p(s)}
\]

Closed-loop poles: roots of \( 1 + KH_p(s) = 0 \)

Example 1 (Speed Control)

\[ x(t): \text{force} \rightarrow M \rightarrow y(t): \text{speed} \]

\[
H_p(s) = \frac{1}{Ms} \quad \rightarrow \quad \text{open-loop pole: } s = 0
\]

Closed-loop pole: \( 1 + K \frac{1}{Ms} = 0 \Rightarrow s = -\frac{K}{M} \)

\[ K \rightarrow \infty \quad K = 0 \]

\[ \text{Im} \quad \text{Re} \quad \text{step response:} \quad r(t) \]

\( \text{larger } K \)
Example 2 (Position Control) \( y(t) : \) position

\[
M \frac{d^2 y}{dt^2} + b \frac{dy}{dt} = x(t)
\]

\[
H_p(s) = \frac{1}{Ms^2 + bs} = \frac{1}{s(Ms + b)}
\]

Open-loop poles: \( s = 0, -\frac{b}{M} \)

Closed-loop poles:

\[
1 + \frac{K}{s(Ms + b)} = 0 \implies Ms^2 + bs + K = 0
\]

\[
s = -\frac{b \pm \sqrt{b^2 - 4KM}}{2M}
\]

**Root-Locus Analysis**

How do the roots of

\[
1 + KH(s) = 0
\]

move as \( K \) is increased from \( K = 0 \) to \( K = +\infty \)?

If a point \( s_0 \in \mathbb{C} \) is on the root locus, then \( H(s_0) = -\frac{1}{K} \) for some \( K > 0 \), therefore \( \angle H(s_0) = \pi \). The rules for sketching the root locus below are derived from this property.

Rules for sketching the root locus:

Let

\[
H(s) = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} = \frac{\prod_{k=1}^{m} (s - \beta_k)}{\prod_{k=1}^{n} (s - \alpha_k)}
\]

\( \beta_k \): zeros \( k = 1, \ldots, m \)

\( \alpha_k \): poles \( k = 1, \ldots, n \)

1) As \( K \to 0 \), the roots converge to the poles of \( H(s) \):

\[
H(s) = -\frac{1}{K} \to \infty
\]

Since there are \( n \) poles, the root locus has \( n \) branches, each starting at a pole of \( H(s) \).
2) As \( K \to \infty \), \( m \) branches approach the zeros of \( H(s) \). If \( m < n \), then 
\( n - m \) branches approach infinity following asymptotes centered at:

\[
\frac{\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{m} \beta_k}{n - m}
\]

with angles:

\[
\frac{180^\circ + (i - 1)360^\circ}{n - m} \quad i = 1, 2, ..., n - m.
\]

Example 2 above: \( n - m = 2 \), poles: 0, \(-b/M\) 
with center = \( -\frac{b}{2M} \), and angles = 90\(^\circ\), -90\(^\circ\)

3) Parts of the real line that lie to the left of an odd number of real poles and zeros of \( H(s) \) are on the root locus. 
Example 1 above: 

4) Branches between two real poles must break away into the complex plane for some \( K > 0 \). The break-away and break-in points can be determined by solving for the roots of \( \frac{dH(s)}{ds} = 0 \)

Example 2 above:

\[
H(s) = \frac{1}{Ms^2 + bs}
\]
\[
\frac{dH}{ds} = \frac{-2Ms - b}{(Ms^2 + bs)^2} = 0 \quad \Rightarrow \quad s = \frac{-b}{2M}
\]

**Example 3:**

\[
H(s) = \frac{s - 1}{(s + 1)(s + 2)}
\]

\(n = 2, \ m = 1\), zeros: \(s = 1\), poles: \(s = 1, -2\).

One asymptote with angle 180°

<table>
<thead>
<tr>
<th>Re</th>
<th>Im</th>
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<tbody>
<tr>
<td>−2</td>
<td>−1</td>
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**Example 4:**

\[
H(s) = \frac{s + 2}{s(s + 1)} \quad n - m = 1 \text{ asymptote with angle } 180°
\]

Break-away / break-in points:

\[
\frac{dH}{ds} = \frac{s^2 + s - (2s + 1)(s + 2)}{s^2(s + 1)^2} = 0
\]

\[
s^2 + s - (2s^2 + 5s + 2) = 0
\]

\[
s^2 + 4s + 2 = 0 \Rightarrow s = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}
\]

**Example 5:**

\[
H(s) = \frac{s + 2}{s(s + 1)(s + a)} \quad a > 2
\]

(Pole at \(-a\) added to the previous example)

\(n - m = 2\), therefore two asymptotes with angles \(\mp 90°\)

Center of the asymptotes: \(\frac{(0 - a) - (-2)}{2} = \frac{1 - a}{2}\)
For large enough $a$, $\frac{dH(s)}{ds} = 0$ has three real, negative roots:

MATLAB command: rltool

High-Gain Instability:
Large feedback gain causes instability if:
1) $H(s)$ has zeros in the right-half plane (nonminimum phase)
2) $n - m \geq 3$

$n - m = 2$  \hspace{1cm}  $n - m = 3$

stable but poorly damped as $K \nearrow$

$n - m = 4$  \hspace{1cm}  $n - m = 5$
\( n - m = 1 \): faster response without losing damping or stability as \( K \uparrow \)

Example: Root locus of a system that can’t be stabilized with constant gain feedback: