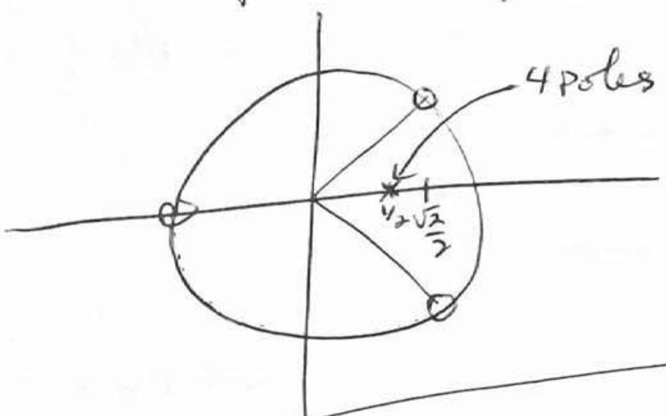
P. Sandborn
PS 12 solns

(1) $H(z)$ s.t. $|H(e^{j0})| = 1$

$$\left.\begin{aligned} |H(e^{j\pi/4})| &= 0 \\ |H(e^{j\pi})| &= 0 \end{aligned}\right\} \text{ 3 zeros}$$

We want LPF, so # poles > # zeros
pick 4 poles:

pole-zero plot



$$\boxed{H(z) = \frac{K\,(z+1)\left(z - \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right)\left(z - \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)}{\left(z - \frac{1}{2}\right)^4}}$$

choose K s.t. $|H(e^{j0})| = 1$,

$$H(1) = \frac{K(2)\left(1 - \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right)\left(1 - \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)}{\left(\frac{1}{2}\right)^4}$$

$$= 32K(2-\sqrt{2}) = 1$$

$$\boxed{K = \frac{1}{32(2-\sqrt{2})}}$$

$$① \quad H(z) = \frac{K(z+1)\left(z-\left(\frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2}\right)\right)\left(z-\left(\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2}\right)\right)}{\left(z-\frac{1}{2}\right)^4}$$

$$Y(z)\left(z-\tfrac{1}{2}\right)^4 = K X(z)(z+1)\left(z-e^{j\pi/4}\right)\left(z-e^{-j\pi/4}\right)$$

$$\left(z-\tfrac{1}{2}\right)^4 = \left(z^2+\tfrac{1}{4}-2z\left(\tfrac{1}{2}\right)\right)^2 = \left(z^2+\tfrac{1}{4}-z\right)^2$$

$$= z^4 + \frac{1}{16} + z^2 + \frac{z^2}{4} - \frac{z}{4} - z^3 + -z^3 + \frac{z^2}{4} - \frac{z}{4}$$

$$= z^4 - 2z^3 + \frac{3}{2}z^2 - \frac{1}{2}z + \frac{1}{16}$$

$$(z+1)\left(z-e^{j\pi/4}\right)\left(z-e^{-j\pi/4}\right) = (z+1)\left(z^2 - ze^{-j\pi/4} - e^{+j\pi/4}z + 1\right)$$

$$= (z+1)\left(z^2 - 2z\cos\left(\tfrac{\pi}{4}\right) + 1\right)$$

$$= z^3 - 2z^2\cos\left(\tfrac{\pi}{4}\right) + z + z^2 - 2z\cos\tfrac{\pi}{4} + 1$$

$$= z^3 + z^2(1-\sqrt{2}) + z(1-\sqrt{2}) + 1$$

$$Y(z)\left(z^4 - 2z^3 + \frac{3}{2}z^2 - \frac{1}{2}z + \frac{1}{16}\right) =$$

$$K X(z)\left(z^3 + (1-\sqrt{2})z^2 + (1-\sqrt{2})z + 1\right)$$

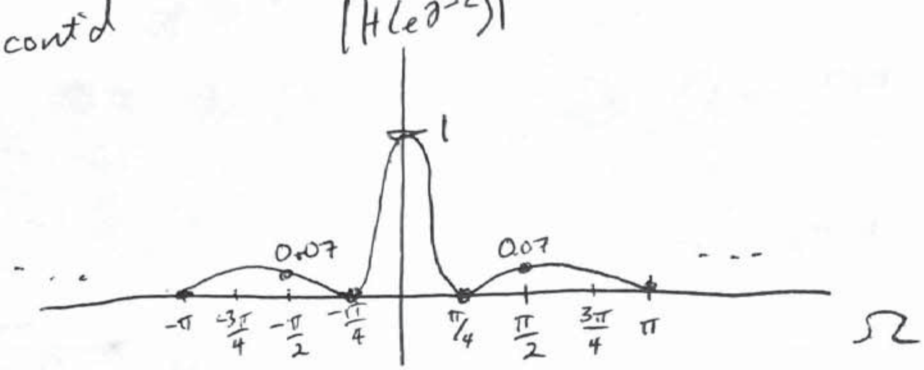
$$y[n+4] - 2y[n+3] + \frac{3}{2}y[n+2] - \frac{1}{2}y[n+1] + \frac{1}{16}y[n]$$

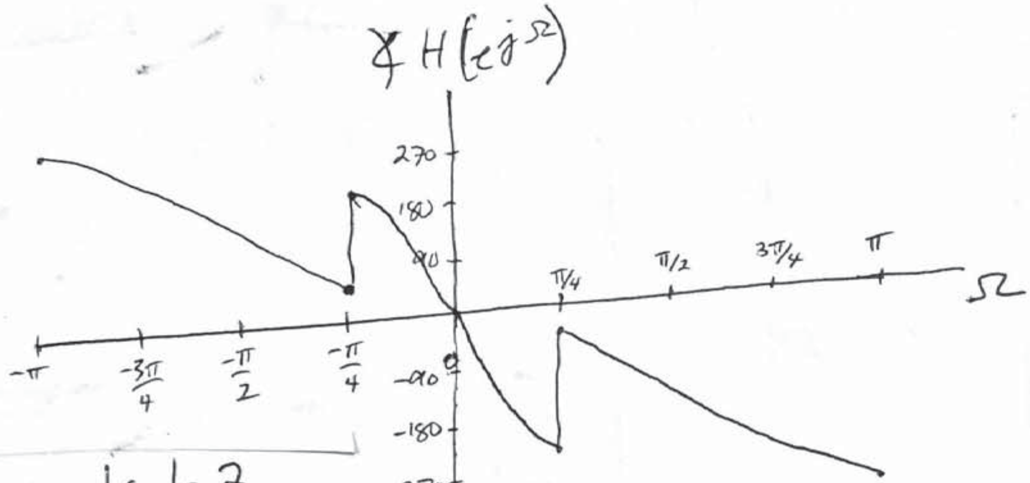
$$= K\left[x[n+3] + (1-\sqrt{2})x[n+2] + (1-\sqrt{2})x[n+1] + x[n]\right]$$

$$\boxed{\begin{aligned} y[n] = {} & Kx[n-1] + K(1-\sqrt{2})x[n-2] + K(1-\sqrt{2})x[n-3] + Kx[n-\cancel{3}4] \\ & +2y[n-1] - \frac{3}{2}y[n-2] + \frac{1}{2}y[n-3] - \frac{1}{16}y[n-4] \end{aligned}}$$

Causal because it doesn't depend on future values of x or y

① cont'd





@ $\Omega = 0$, $|r_{z1}| = 2$

$|r_{z2}| = \sqrt{2-\sqrt{2}}$

$|r_{z3}| = \sqrt{2-\sqrt{2}}$

$$|H(e^{j(\Omega=0)})| = \frac{1}{32(2-\sqrt{2})}\frac{(2)(2-\sqrt{2})}{1/2^4} = 1 \checkmark$$

@ $\Omega = \pi/2$, $|r_{z1}| = \sqrt{2}$ $\qquad |r_{p1}| = \frac{\sqrt{5}}{2}$

$|r_{z2}| = \sqrt{2-\sqrt{2}}$

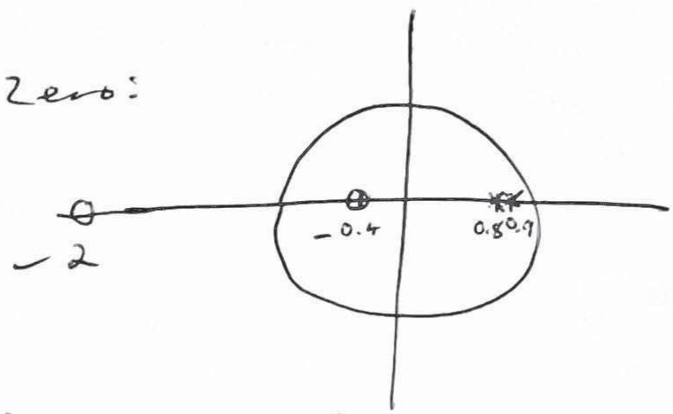
$|r_{z3}| = \sqrt{2+\sqrt{2}}$

$$|H(e^{j\pi/2})| = \frac{1}{32(2-\sqrt{2})}\frac{\sqrt{2}\sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2}}}{25/16} = \frac{\sqrt{2}}{2(2-\sqrt{2})}\frac{\sqrt{4-2}}{25} = \frac{1}{25(2-\sqrt{2})} = 0.07$$

② $$H(z) = \frac{(1+0.4z^{-1})(1+2z^{-1})}{(1-0.9z^{-1})(1-0.8z^{-1})}$$

$$H(z)\frac{z^2}{z^2} = \frac{(z+0.4)(z+2)}{(z-0.9)(z-0.8)}$$

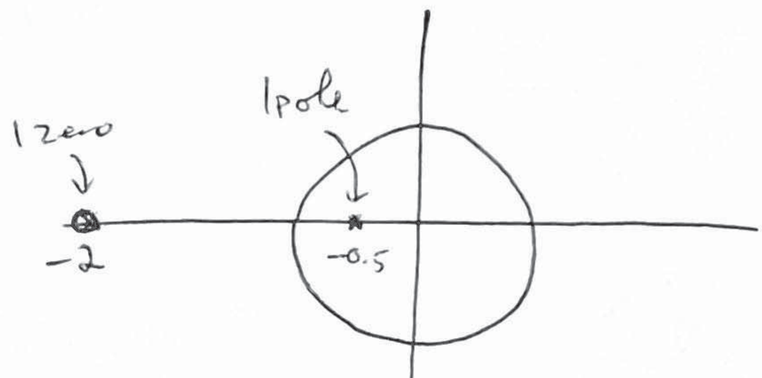
a) pole zero:



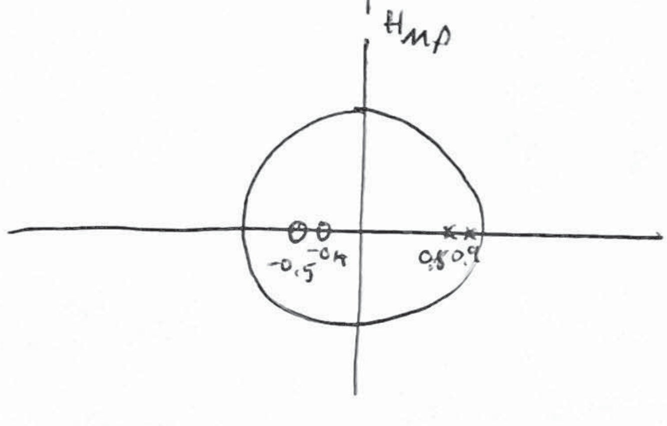
system is stable

b) Find H_{AP} and H_{MP} s.t. $H(z) = H_{AP}\, H_{MP}$

H_{AP} pole-zeros



$$\boxed{H_{AP} = \frac{(z+2)}{(z+0.5)}}$$



$$\cancel{H_{MP} = \frac{(z-0.8)(z-0.9)}{}}$$

$$\boxed{H_{MP} = \frac{(z+0.5)(z+0.4)}{(z-0.8)(z-0.9)}}$$

2c) Want H_{inv} stable/causal s.t. $|H_{inv}||H| = 1 \ \forall \Omega$

$$|H_{inv} H| = |H_{inv} H_{AP} A_{MP}| = |H_{inv} H_{MP}|$$

$$\text{choose } H_{inv} = H_{MP}^{-1} = \boxed{\frac{(z-0.8)(z-0.9)}{(z+0.5)(z+0.4)} = H_{inv}(z)}$$

stable, causal.

3 a)

$$p[n+1] = 1.05\,p[n] - 0.15\,s[n]$$

$$s[n+1] = s[n]\,1.03$$

First, $S(z)$?

$$1.03\,S(z) = z\,S(z) - z\,s[0]$$

$$(1.03 - z)\,S(z) = -z\,s[0]$$

$$S(z) = \frac{z\,s[0]}{z-1.03}$$

$$S(z) \longleftrightarrow \underline{s[n] = s[0](1.03)^n u[n]}$$

plug into $p[n]$, find $P(z)$

$$z\left(P(z) - p[0]\right) = 1.05\,P(z) - 0.15\,s[0]\,\frac{z}{z-1.03}$$

So: $$\boxed{P(z) = \frac{p[0]\,z - 0.15\,s[0]\,\frac{z}{z-1.03}}{z-1.05}}$$

$$P(z) = \frac{p[0]\,z(z-1.03) - 0.15\,s[0]\,z}{(z-1.03)(z-1.05)}$$

$$P(z) = \frac{p[0]\,z^2 + \left(-1.03\,p[0] - 0.15\,s[0]\right)z}{(z-1.03)(z-1.05)}$$

Let $A_1 = p[0]$, $A_2 = -1.03\,P(0) - 0.15\,s[0]$

$$P(z) = \frac{A_1 z^2 + A_2 z}{(z-1.03)(z-1.05)} = \frac{B_1}{z-1.03} + \frac{B_2}{z-1.05} + B_3$$

plug in $p[0] = 500{,}000$, $s[0] = 100{,}000$

Find: $B_1 = 772{,}500$

$B_2 = -262{,}500$

$B_3 = 500{,}000$

$$P(z) = \frac{772{,}500}{z-1.03} - \frac{262{,}500}{z-1.05} + 500{,}000$$

$\mathcal{Z}^{-1}$

$$\boxed{p[n] = \left(772{,}500\,(1.03)^{n-1} - 262{,}500\,(1.05)^{n-1}\right)u(n-1) + 500{,}000\,\delta[n]}$$

(3) cont'd

$$0 = B_1 (1.03)^{n-1} - B_2 (1.05)^{n-1} =$$

$$B_1 (1.03)^{n-1} = B_2 (1.05)^{n-1}$$

$$\frac{B_1}{B_2} = \left(\frac{1.05}{1.03}\right)^{n-1}$$

$$\log_{10} \frac{B_1}{B_2} = (n-1) \log_{10} (1.05/1.03)$$

$$n = 1 + \frac{\log_{10}(B_1/B_2)}{\log_{10}(1.05/1.03)} = 57.12$$, so loan is paid $\boxed{\text{off after 58 years}}$

total paid? $s[n] = s[0]\, 1.03^n u[n]$

amount paid in year n: $x[n] = 0.15 s[n] = s[0](1.03)^n (0.15)$

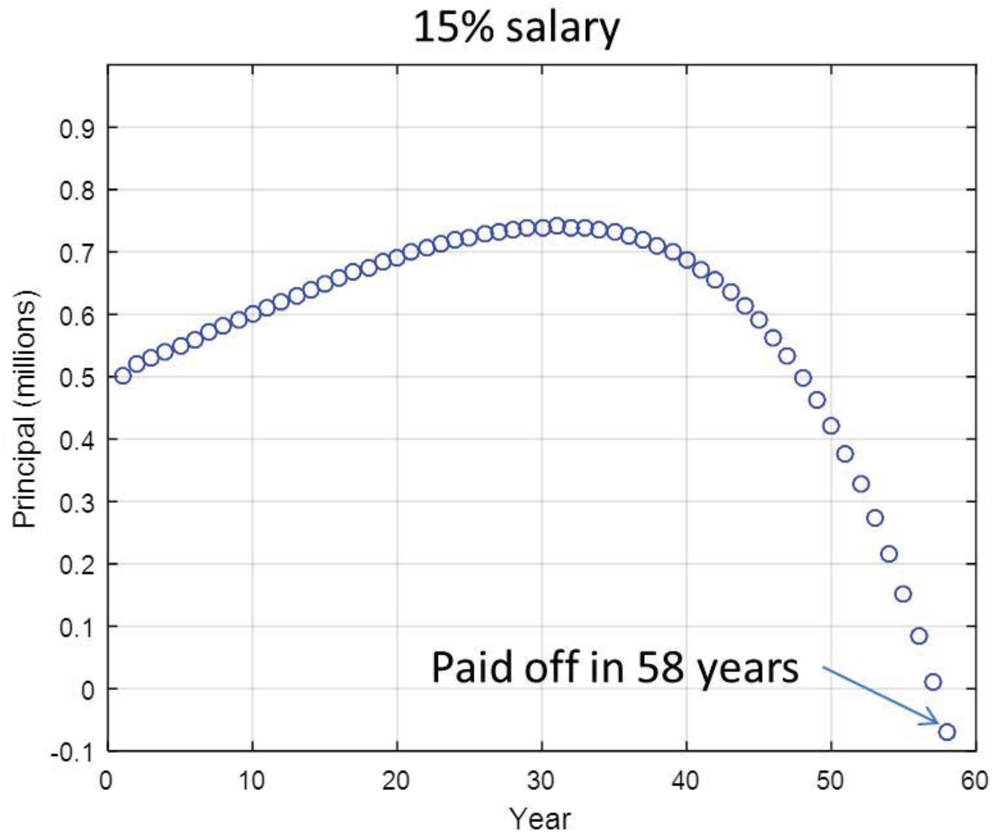
total paid up to year n: $$\sum_{l=1}^{n} x[l] = \sum_{l=1}^{n} s[0](0.15)(1.03)^l$$

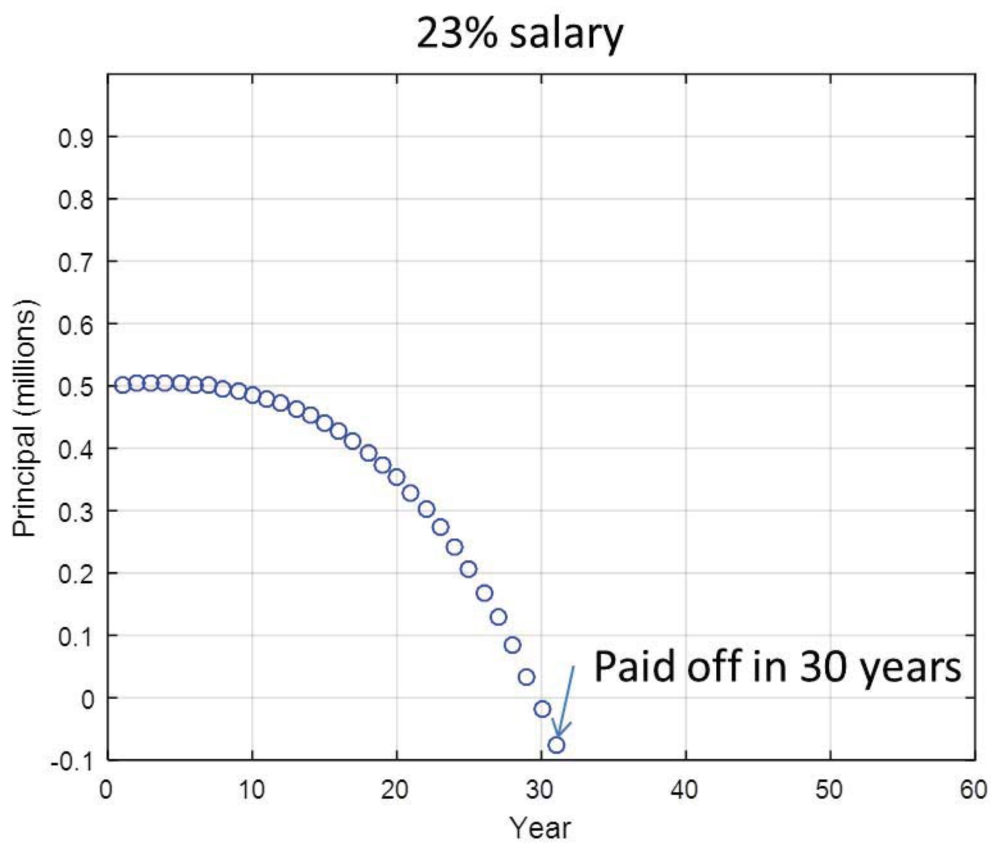
$$\sum_{l=1}^{58} x[l] = s[0](0.15) \frac{(1.03)(1 - 1.03^{58})}{1 - 1.03}$$

$$= s[0](0.15)(1.03)(151.8) = \$2.35 \times 10^6$$

$\cancel{11.7e6 = \$11.7 \text{ million}}$

$\boxed{\text{total paid} = \$2.35 \text{ million}}$


15% salary
Principal (millions)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
10
20
30
40
50
60
Year
Paid off in 58 years


23% salary
Principal (millions)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
10
20
30
40
50
60
Year
Paid off in 30 years

(4)

$$G(s) = \frac{500\,(s+0.5)}{s^2(s+10)^2}$$

$$E = R - Y \qquad Y = EG$$

$$E = R - EG \qquad E(1+G) = R \;,\; E = \frac{R}{1+G}$$

a) 2 poles @ $s=0$ for G, $\therefore$ $\boxed{\text{Type 2}}$ system

b) $\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s)$? constant.

$$\cancel{sE(s) = \frac{R(s)\, s^2(s+10)^2}{s^2(s+10)^2 + 500(s+0.5)}}$$

$$sE(s) = s\frac{R(s)}{1+\frac{500(s+0.5)}{s^2(s+10)^2}} = s\,\frac{R(s)\, s^2(s+10)^2}{s^2(s+10)^2 + 500(s+0.5)}$$

Need $R(s) = \frac{1}{s^3}$, so numerator → constant
and denominator → constant

$$\boxed{R(s) = \frac{1}{s^3} \xleftrightarrow{\mathcal{L}^{-1}} \boxed{t^2 u(t) = r(t)}}$$

c) $\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{(s+10)^2}{500(s+0.5)} = \frac{100}{250} = \boxed{\frac{2}{5} = \lim_{t\to\infty} e(t)}$

(5) $E = \frac{R}{1+DG} - W\frac{G}{1+GD} = E_R(s)R(s) - E_W(s)W(s)$

a) $r(t)$ = unit step, what's $\lim_{t\to\infty} e_R(t)$?

$R(s) = \frac{1}{s}$

$$\lim_{t\to\infty} e_R(t) = \lim_{s\to 0} sE_R(s) = \lim_{s\to 0} \frac{s}{s}\frac{1}{1+\frac{k_p s + k_I}{s}\frac{1}{s(ms+b)}}$$

$$= \lim_{s\to 0} \frac{s}{s + \frac{k_p s + k_I}{s(ms+b)}}$$

$$= \lim_{s\to 0} \frac{s^2(ms+b)}{s^2(ms+b) + k_p(s) + k_I}$$

$$\boxed{\lim_{t\to\infty} e_R(t) = \frac{0}{0+k_I} = \frac{0}{k_I} \to 0}$$

(5)

6) if $w(t) = 0.01 \sin(4t) = A_w \cos(4t + \phi_w)$

to find pk-topk error, we use assumption for LTI basis functions,

$$e(t) = A_e \cos(4t + \phi_e)$$

$$A_e = |E_w(j4)|\, A_w$$

$$E_w(s) = \frac{\frac{1}{(ms+b)s}}{1 + \frac{k_p s + k_I}{s}\frac{1}{s(ms+b)}} = \frac{s}{ms^3 + bs^2 + k_p s + k_I}$$

if $m = 1,\ b = 10,\ k_p = 30,\ k_I = 30$

$$E_w(s) = \frac{s}{s^3 + 10s^2 + 30s + 30}$$

$$E_w(j\omega) = \frac{j\omega}{-j\omega^3 - 10\omega^2 + j30\omega + 30}$$

$$E_w(j4) = \frac{4j}{-64j - 160 + 120j + 30} = \frac{4j}{-130 + 56j}$$

$$|E_w(4j)| = \frac{4}{\sqrt{130^2 + 56^2}} = \frac{4}{141.5} = 0.028$$

$$\boxed{A_e = 0.028 \times 0.01 = 0.28 \times 10^{-3}}$$

(5c)

If $D(s) = \frac{k_p s + k_I}{s} \frac{100(s^2+2s+17)}{s^2+16}$

We know that $E(s) = W(s) \frac{G}{1+DG}$

$G = \frac{1}{s(ms+b)}$ D as above,

$$\frac{E(s)}{W(s)} = \frac{\frac{1}{s(ms+b)}}{1 + \frac{k_p s + k_I}{s} \frac{100(s^2+2s+17)}{(s^2+16)} \frac{1}{(ms+b)s}}$$

$$\frac{E}{W} = \frac{s(s^2+16)}{s(s^2+16)(ms+b) + (k_p s + k_I)(100)(s^2+2s+17)}$$

plug in $j\omega = j4$

$$\frac{E}{W} = \frac{0}{0 + (k_p(j4) + k_I)(100)(-4+8j+17)}$$

$$\boxed{\left.\frac{E}{W}\right|_{@\omega=4} = 0}$$

$\boxed{pk\ to\ pk = 0}$