# EE120: Signals and Systems

## Problem Set 4 Solutions

## 9/23/2016

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(I) i) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad C = [-\frac{1}{2}, 1, \frac{1}{2}]$, $D = 1$

$$\bar{x}[n+1] = A\bar{x}[n] + B u[n]$$

$$\begin{bmatrix} x[n-2] \\ x[n-1] \\ x[n] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x[n-3] \\ x[n-2] \\ x[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[n]$$

$x[n-2] = x[n-2]$ ✓

$x[n-1] = x[n-1]$ ✓

$x[n] = x[n-3](-\frac{1}{2}) + x[n-2](1) + x[n-1](\frac{1}{2}) + u[n]$

Rearranging: $u[n] = x[n] + (\frac{1}{2})x[n-3] - x[n-2] - (\frac{1}{2})x[n-1]$

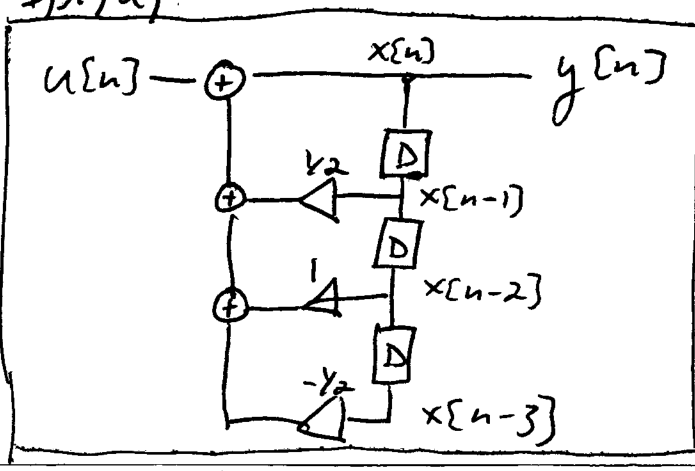
$y[n] = C\bar{x}[n] + Du[n]$

~~$y[n] = (-\frac{1}{2}x[n]$~~ $\quad y[n] = [-\frac{1}{2}, 1, \frac{1}{2}] \begin{bmatrix} x[n-3] \\ x[n-2] \\ x[n-1] \end{bmatrix} + u[n]$

$$y[n] = \left(-\frac{1}{2}\right) x[n-3] + (1)x[n-2] + \left(\frac{1}{2}\right)x[n-1] + u[n]$$

$u[n] = x[n] + \frac{1}{2}x[n-3] - x[n-2] - \frac{1}{2}x[n-1]$

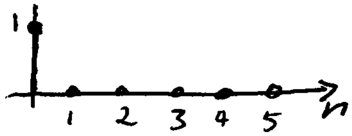
so: $y[n] = x[n]$ 1)i)a)



$\left(\frac{1}{2}\right) y[n-1] + (1) y[n-2] + \left(-\frac{1}{2}\right) y[n-3] + u[n] = y[n]$

LDE from block Diagram 1) i) b)

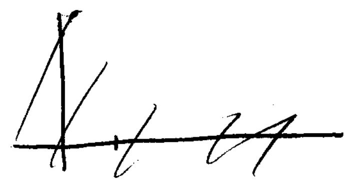
$$\boxed{y[n] - \left(\frac{1}{2}\right) y[n-1] - (1) y[n-2] + \frac{1}{2} y[n-3] = u[n]}$$

$u[n] =$ 

1) i) c)

$$y[n] = u[n] + \frac{1}{2} y[n-1] + y[n-2] - \frac{1}{2} y[n-3]$$

| $n$ | $u[n]$ | $y[n]$ |
|---|---|---|
| 0 | 1 | $\boxed{1}$ |
| 1 | 0 | $\boxed{\frac{1}{2}}$ |
| 2 | 0 | $\frac{1}{2} \cdot \frac{1}{2} + 1 = \boxed{\frac{5}{4}}$ |
| 3 | 0 | $\frac{5}{4} \cdot \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \boxed{\frac{5}{8}}$ |
| 4 | 0 | $\frac{5}{8} \cdot \frac{1}{2} + \frac{5}{4} - \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{21}{16}}$ |
| 5 | 0 | $\frac{21}{16} \cdot \frac{1}{2} + \frac{5}{8} - \frac{1}{2} \cdot \frac{5}{4} = \boxed{\frac{21}{32}}$ |



① ii) $A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ $B=\begin{bmatrix}0\\0\\1\end{bmatrix}$ $C=\left[\frac{1}{2}, -1, -\frac{1}{2}\right]$ $D=1$

$\bar{x}[n+1] = A\bar{x}[n] + Bu[n]$

$$\begin{bmatrix}x[n-2]\\x[n-1]\\x[n]\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x[n-3]\\x[n-2]\\x[n-1]\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u[n]$$

assigns $x[n-2] = x[n-2]$ ✓

$x[n-1] = x[n-1]$ ✓

$\boxed{x[n] = 0 + u[n]}$

$y[n] = C\bar{x}[n] + Du[n]$

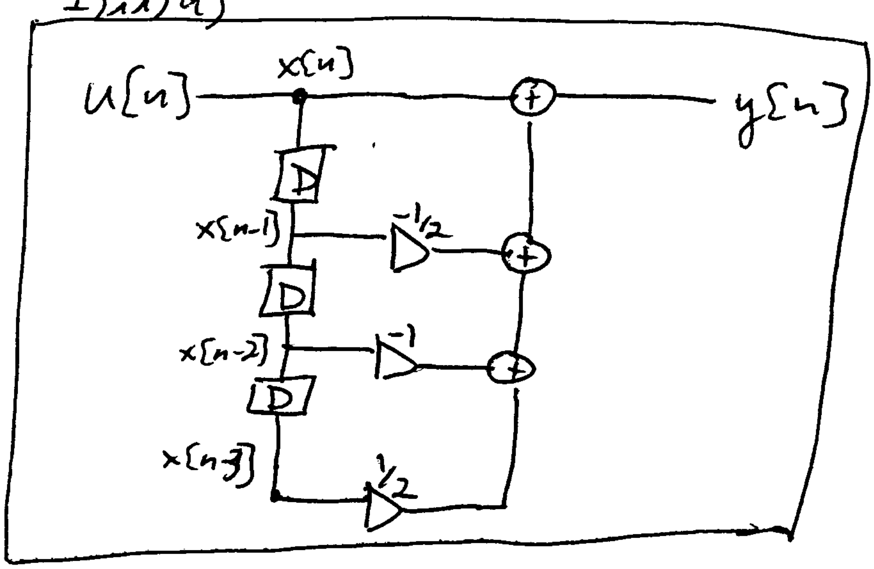
$$y[n] = \left[\frac{1}{2}, -1, -\frac{1}{2}\right]\begin{bmatrix}x[n-3]\\x[n-2]\\x[n-1]\end{bmatrix} + u[n]$$

$y[n] = \frac{1}{2}x[n-3] - x[n-2] - \frac{1}{2}x[n-1] + u[n]$

$y[n] = \frac{1}{2}x[n-3] - x[n-2] - \frac{1}{2}x[n-1] + x[n]$

1) ii) a)

so:



$$y[n] = -\frac{1}{2}u[n-1] - u[n-2] + \frac{1}{2}u[n-3] + u[n]$$

LDE from block Diagram 1)ii)b)

$$\boxed{y[n] = u[n] + \left(-\frac{1}{2}\right)u[n-1] + (-1)u[n-2] + \left(\frac{1}{2}\right)u[n-3]}$$

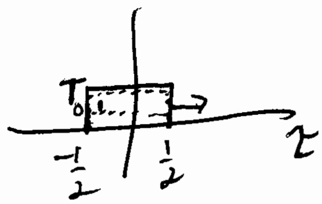
1)ii)c)

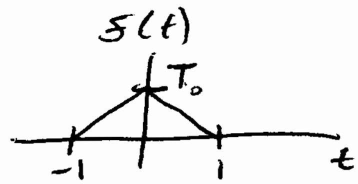
| $n$ | $u[n]$ | $y[n]$ |
|---|---|---|
| 0 | 1 | $\boxed{1}$ |
| 1 | 0 | $\boxed{-\frac{1}{2}}$ |
| 2 | 0 | $\boxed{-1}$ |
| 3 | 0 | $\boxed{\frac{1}{2}}$ |
| 4 | 0 | $\boxed{0}$ |
| 5 | 0 | $\boxed{0}$ |

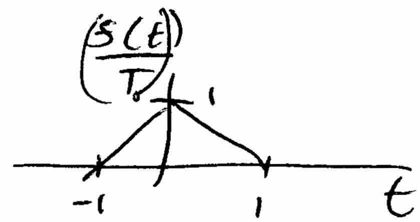
Problem 2: In general, we have form:

$$x(t) = T_0 \cdot \Pi(t) * \Pi(t) * \frac{1}{T_0}\text{comb}\left(\frac{t}{T_0}\right) = \Pi(t) * \Pi(t) * \text{comb}\left(\frac{t}{T_0}\right)$$

$s(t) = T_0 \Pi(t) * \Pi(t)$ :

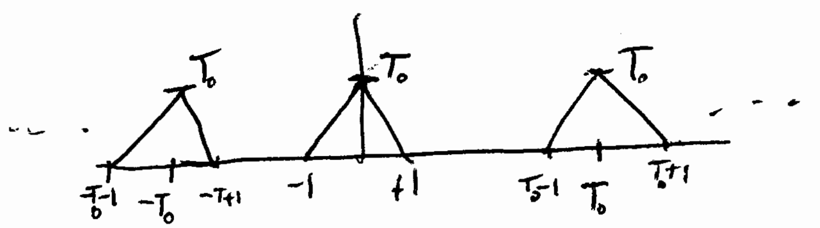



$$s(t) = \begin{cases} -T_0 t + T_0 & 0 < t < 1 \\ T_0 t + T_0 & -1 < t < 0 \end{cases}$$

$x(t) = s(t) * \frac{1}{T_0}\text{comb}\left(\frac{t}{T_0}\right)$ # vertically scaled triangle function, spaced @ interval $T_0$




Fund. period is $T_0$

Fund. freq $\omega_0 = \frac{2\pi}{T}$

Fourier coeffs are: $a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\frac{2\pi}{T_0}t}\, dt$

$$a_k = \frac{1}{T_0}\left[\int_{-1}^{0} (T_0 t + T_0) e^{-jk\frac{2\pi}{T_0}t}\, dt + \int_{0}^{1} (-T_0 t + T_0) e^{-jk\frac{2\pi}{T_0}t}\, dt\right]$$

$$a_k = \int_{-1}^{0} t\, e^{-j\omega_0 k t}\, dt + \int_{0}^{1} -t\, e^{-j\omega_0 k t}\, dt + \int_{-1}^{1} e^{-jk\omega_0 t}\, dt$$

PS4 (2)

$$\int_{-1}^{0} t e^{-jk\omega_0 t} dt = e^{-jk\omega_0 t}\left(\frac{-jk\omega_0 t - 1}{-k^2 \frac{4\pi^2}{T_0^2}}\right)\Bigg|_{-1}^{0} = \frac{1}{k^2\omega_0^2} + \frac{1}{k^2\omega_0^2} e^{jk\omega_0}(jk\omega_0) ||$$

$$@0, \quad = 1\left(\frac{-1}{-k^2\omega_0^2}\right) = \frac{1}{k^2\omega_0^2}$$

$$@-1, \quad e^{jk\omega_0}\left(\frac{jk\omega_0 - 1}{-k^2\omega_0^2}\right)$$

$$-\int_{0}^{1} t e^{-jk\omega_0 t} dt = e^{-jk\omega_0 t}\frac{(jk\omega_0 t + 1)}{-k^2\omega_0^2}\Bigg|_{0}^{1}$$

$$= \frac{e^{-jk\omega_0}}{-k^2\omega_0^2}(jk\omega_0 + 1) + 1\left(\frac{1}{+k^2\omega_0^2}\right)$$

$$\int_{-1}^{1} e^{-jk\omega_0 t} dt = \frac{2\sin(k\omega_0)}{k\omega_0}$$

$$\text{so, } a_k = \frac{1}{k^2\omega_0^2}\left[1 + e^{jk\omega_0} jk\omega_0 - e^{jk\omega_0} - e^{-jk\omega_0} jk\omega_0 - e^{-jk\omega_0} + 1\right] + \frac{2\sin(k\omega_0)}{k\omega_0}$$

$$a_k = \frac{1}{k^2\omega_0^2}\left[2 + jk\omega\left(2j\sin(k\omega_0)\right) - 2\cos(k\omega_0)\right] + \frac{2\sin(k\omega_0)}{k\omega_0}$$

$$a_k = \frac{2}{k^2\omega_0^2}\left(1 + -k\omega\sin(k\omega_0) - \cos(k\omega_0)\right] + \frac{2\sin(k\omega_0)}{k\omega_0}$$

$$a_k = \frac{1}{k\omega_0}\left(\frac{2}{k\omega_0} - \cancel{2\sin(k\omega_0)} - \frac{2\cos(k\omega_0)}{k\omega_0}\right] + \cancel{\frac{2\sin(k\omega_0)}{k\omega_0}} = \frac{2}{k^2\omega_0^2}\left(1 - \cos(k\omega_0)\right)$$

PS4 / 2

$$\frac{2}{k^2\omega_0^2}\frac{2}{2}\left(1-\cos(k\omega_0)\right) = \frac{4}{k^2\omega_0^2}\left(\frac{1-\cos\left(2\left(\frac{k\omega_0}{2}\right)\right)}{2}\right)$$

$$= \frac{4}{k^2\omega_0^2}\left(\sin^2\left(\frac{k\omega_0}{2}\right)\right)$$

$$\omega_0 = \frac{2\pi}{T_0}, \quad a_k = \frac{4}{\frac{k^2 4\pi^2}{T_0^2}} \sin^2\left(k\frac{2\pi}{2T_0}\right)$$

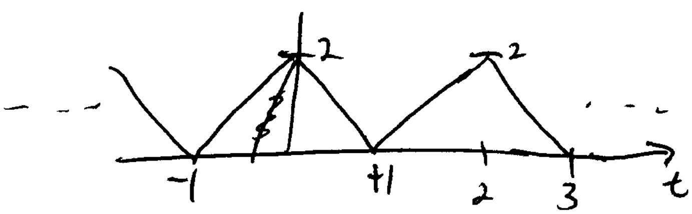
$$a_k = \frac{T_0^2}{k^2\pi^2}\sin^2\left(k\frac{\pi}{T_0}\right)$$

$$\boxed{a_k = \left(\frac{T_0 \sin\left(k\frac{\pi}{T_0}\right)}{k\pi}\right)^2}$$

$$a_0 = \frac{T_0\frac{\pi}{T_0}}{\pi} = 1$$

SO:

2a) $x_1(t)$

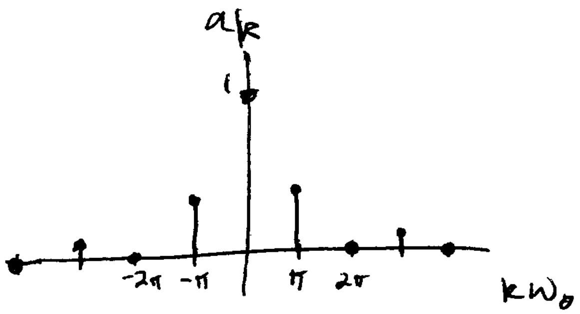



$$\text{so:} \quad a_k = \left(\frac{2\sin\left(k\frac{\pi}{2}\right)}{k\pi}\right)^2$$

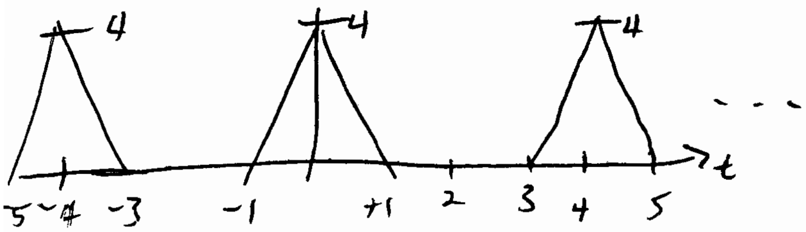
$T_0 = 2 \quad \omega_0 = \frac{2\pi}{2} = \pi$

$$\boxed{a_k = \frac{4\sin^2\left(k\frac{\pi}{2}\right)}{k^2\pi^2}}$$

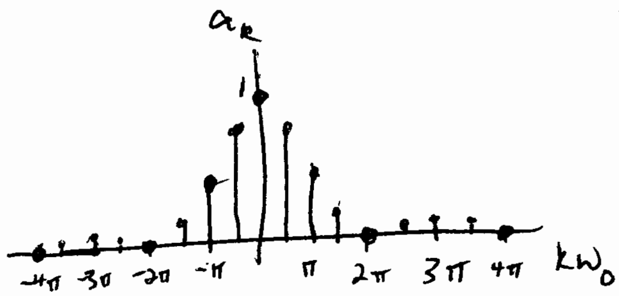
$a_0 = 1$



2b) $x_2$, $T_0 = 4$ $\quad \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$




$T_0 = 4$, so $\quad a_k = \left(\frac{4\sin(k\frac{\pi}{4})}{k\pi}\right)^2 = \boxed{\frac{16\sin^2(k\frac{\pi}{4})}{k^2\pi^2}}$

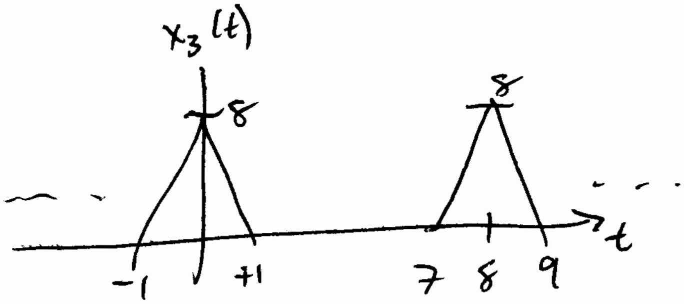


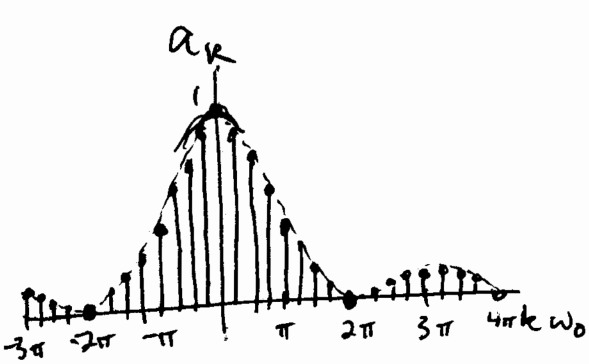

2c) $x_3$, $T_0 = 8$ $\quad \omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

$T_0 = 8$, so,

$$a_k = \left(\frac{8\sin(k\frac{\pi}{8})}{k\pi}\right)^2$$

$$\boxed{a_k = \frac{64\sin^2(k\frac{\pi}{8})}{k^2\pi^2}}$$



2d) $a_k$ for part a): $a_k = \dfrac{4\sin^2(k\frac{\pi}{2})}{k^2\pi^2}$

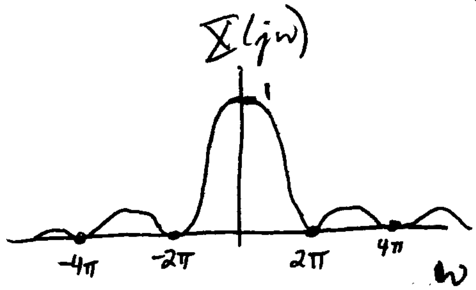
b): $a_k = \dfrac{16\sin^2(k\frac{\pi}{4})}{k^2\pi^2}$

c) $a_k = \dfrac{64\sin^2(k\frac{\pi}{8})}{k^2\pi^2}$

~~4) sketch~~

$\mathcal{F}\{\Pi(t) * \Pi(t)\} = \mathcal{F}\{\Pi(t)\} \cdot \mathcal{F}\{\Pi(t)\}$

$$X(j\omega) = \left(\frac{2\sin(\omega\frac{1}{2})}{\omega}\right)^2 = \frac{4\sin^2(\frac{\omega}{2})}{\omega^2}$$




L'Hopital: @$\omega = 0$ $\dfrac{4\sin(\frac{\omega}{2})\cos(\frac{\omega}{2})}{2\omega} \to 1$

$\boxed{X(j\omega) \text{ is envelope for } a_k\text{'s when } a_k \text{ plotted versus } \underline{k\omega}}$

3a) Consider first, $y = h(t) * \Pi(t)$

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} u(t-\tau)\, \Pi(\tau)\, d\tau$$

$-\frac{1}{2} < \tau < \frac{1}{2}$ ← upper and lower bound on $\tau$

$t - \tau > 0$

$t > \tau$ ← upper integration bound on $\tau$

$$(*) \quad y(t) = \int_{-1/2}^{t} e^{-(t-\tau)}\, d\tau \qquad \text{for } -\tfrac{1}{2} < t < \tfrac{1}{2}$$

$$(**) \quad y(t) = \int_{-1/2}^{1/2} e^{-(t-\tau)} (d\tau) \qquad \text{for } t > \tfrac{1}{2}$$

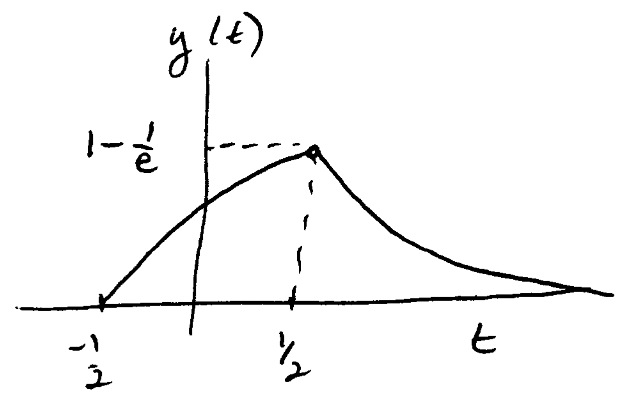
$$(***) \quad y(t) = 0 \quad \text{for } t < -\tfrac{1}{2}$$

$$(*): \quad y(t) = e^{-t} \int_{-1/2}^{t} e^{\tau}\, d\tau = e^{-t} \left. e^{\tau} \right|_{-1/2}^{t} = e^{-t}\left(e^{t} - e^{-1/2}\right)$$

$$(**): \quad y(t) = e^{-t} \int_{-1/2}^{1/2} e^{\tau}\, d\tau = e^{-t} \left. e^{\tau} \right|_{-1/2}^{1/2} = e^{-t}\left(e^{1/2} - e^{-1/2}\right)$$

$$\boxed{y(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 - e^{-(t+1/2)} & -\frac{1}{2} < t < \frac{1}{2} \\ e^{-t+1/2} - e^{-t-1/2} & t > \frac{1}{2} \end{cases}}$$

$$1 - e^{-1} = y(\tfrac{1}{2}) = 1 - \tfrac{1}{e}$$



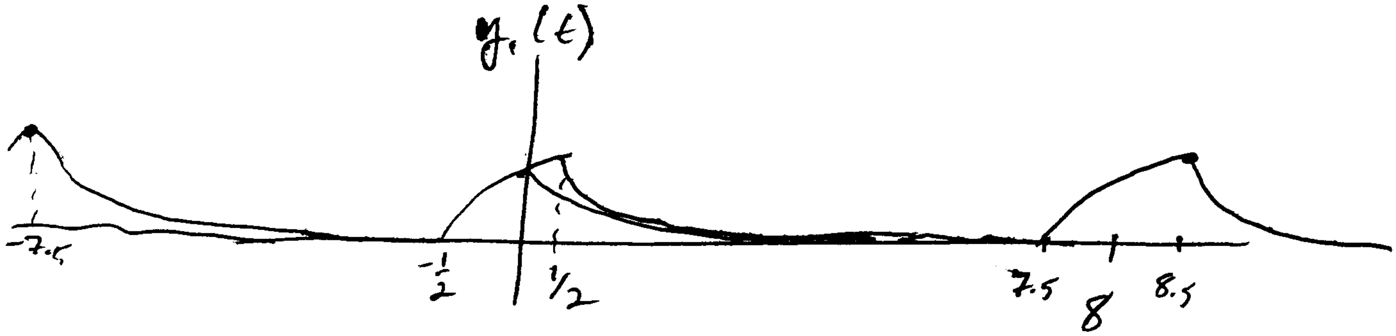
To get $y_1(t) = y(t) * \frac{1}{8}\text{comb}\left(\frac{t}{8}\right)$

This maybe complicated for analysis, because $y(t)$ is bound on left by $-\frac{1}{2}$, but unbounded on right. However, the comb has period 8, so the contribution of adjacent $y(t)$'s is on the order of $e^{-8+\frac{1}{2}} - e^{-8-\frac{1}{2}} = e^{-7.5} - e^{-8.5}$, which is <u>very small</u>.

so, we can define $\tilde{y}(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 - e^{-(t+\frac{1}{2})} & -\frac{1}{2} < t < \frac{1}{2} \\ e^{-t+\frac{1}{2}} - e^{-t-\frac{1}{2}} & \frac{1}{2} < t < 7.5 \end{cases}$

approx. $y(t)$

The final convolution will be plotted as:



3b) $\mathcal{F}(x(t)) = X(j\omega)$

$$x(t) = \Pi(t) * \frac{1}{8}\,\text{comb}\left(\frac{t}{8}\right) = \frac{1}{8}\cdot\left(\Pi(t) * \text{comb}\left(\frac{t}{8}\right)\right)$$

$$X(j\omega) = \frac{1}{8}\,\mathcal{F}(\Pi(t))\,\mathcal{F}\left(\text{comb}\left(\frac{t}{8}\right)\right)$$

$$= \frac{1}{8}\,\frac{2\sin(\omega\frac{1}{2})}{\omega}\cdot\mathcal{F}\left(\cancel{\text{comb}}\sum_n \delta\left(\frac{t}{8}-n\right)\right)$$

$$= \frac{2}{8}\,\frac{\sin(\omega\frac{1}{2})}{\omega}\cdot\sum_n \mathcal{F}\left[\delta\left(\frac{1}{8}(t-8n)\right)\right]$$

$$= \frac{1}{4\omega}\sin\left(\frac{\omega}{2}\right)\sum_n \mathcal{F}\left[8\delta[t-8n]\right]$$

$$= \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)\sum_n \mathcal{F}[\delta(t-8n)) = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)\mathcal{F}\left[\sum_n \delta(t-8n)\right]$$

$$\cancel{= \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)\sum_n e^{-j\omega 8n}\,\mathcal{F}\{\delta(t)\}}$$

$$\cancel{= \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)\sum_{n=-\infty}^{\infty} e^{-j\omega 8n}} = \cancel{\frac{\sin(\frac{\omega}{2})}{\omega}\left(1+\sum_{n=1}\cos(\omega n)\right)}$$

$$\cancel{e^0 + e^{-8j\omega} + e^{8j\omega} + e^{-16j\omega} + e^{16j\omega} + \vdots = 1 + 2\cos(8\omega) + 2\cos(16\omega) + \dots = 1 + 2\sum_{n=1}\cos(\omega n)}$$

$$\frac{2}{w}\sin\left(\frac{w}{2}\right)F\left[\sum_n \delta(t-8n)\right] = \frac{2}{w}\sin\left(\frac{w}{2}\right)\left(\frac{2\pi}{8}\sum_{k=-\infty}^{\infty}\delta\left(w-\frac{2\pi k}{8}\right)\right)$$

$$= \frac{2\pi\cdot 2}{8w}\sin\left(\frac{w}{2}\right)\sum_{k=-\infty}^{\infty}\delta\left(w-\frac{\pi k}{4}\right)$$

$$\boxed{X(jw) = \frac{\pi}{2w}\sin\left(\frac{w}{2}\right)\sum_{-\infty}^{\infty}\delta\left(w-\frac{\pi}{4}k\right)}$$

$$H(jw) = F(h(t)) = \underbrace{F(e^{-t}u(t))}_{\text{Table 4.2}} = \boxed{\frac{1}{1+jw}}$$

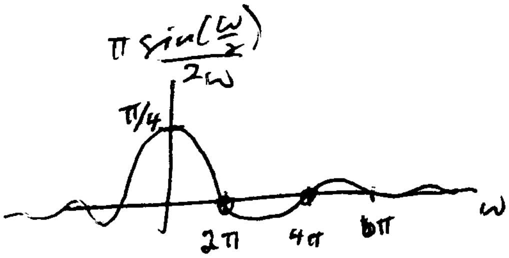
$$\boxed{Y_2(jw) = H(jw)X(jw) = \frac{\pi}{2w}\sin\left(\frac{w}{2}\right)\left(\sum_{k=\infty}^{\infty}\delta\left(w-\frac{\pi}{4}k\right)\right)\frac{1}{1+jw}}$$

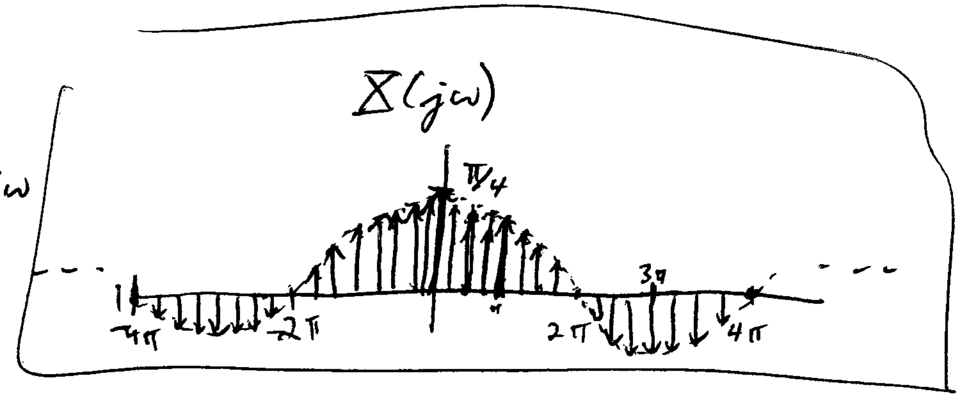
3c) $\frac{\pi \sin\left(\frac{\omega}{2}\right)}{2\omega} = 0$ @ $\frac{\omega}{2} = n\pi$

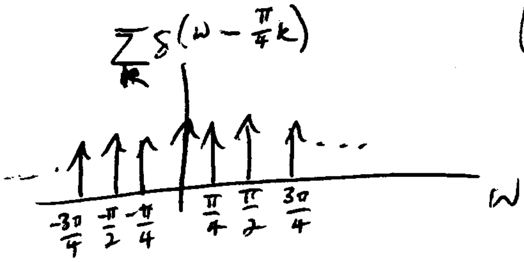
$\omega = 2\pi n$

@ $\omega = 0$, $\frac{\pi \sin\left(\frac{\omega}{2}\right)}{2\omega} \rightarrow \frac{\pi}{2} \frac{\cancel{\cos}\left(\frac{\omega}{2}\right)\frac{1}{2}}{1}$

$= \frac{\pi}{4}$



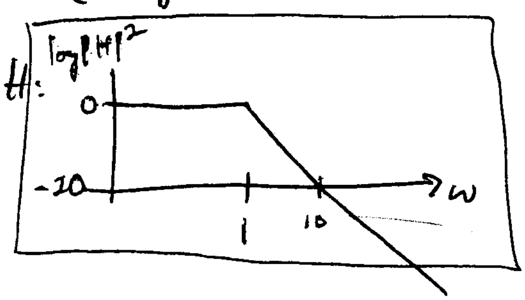




$|H(j\omega)|$?

$H(j\omega) = \frac{1}{1+j\omega} \frac{(1-j\omega)}{(1-j\omega)} = \frac{1-j\omega}{1+\omega^2}$

Bode plot of H:

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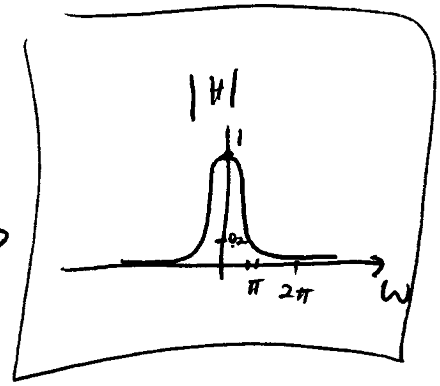
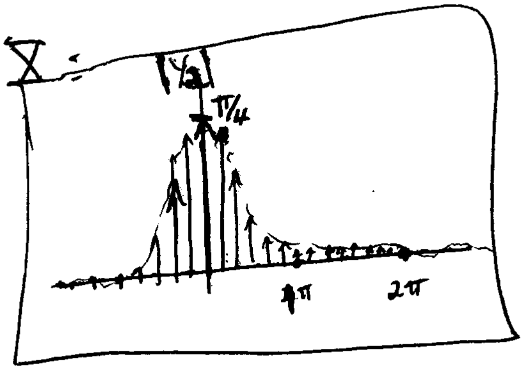



$Y_2 = H X =$



3d) Fourier Series of $y_1(t)$ is derived:

1st Step: $x(t) = \sum_k a_k e^{+j\omega_0 k t}$

Fourier series of periodic box.

2nd Step: $y_1(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$

Integrand and sum can be manipulated to have the form:

$$y_1(t) = \sum_k b_k e^{+j\omega_0 k t}$$

and $b_k$ will be Fourier coeffs of $y_1$.

1st Step: $a_k = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jk\frac{2\pi}{T_0}t} dt$ $\qquad T_0 = 8$, $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{4}$

$$a_k = \frac{1}{8} \int_{-1/2}^{1/2} e^{-jk\frac{2\pi}{8}t} dt = \frac{1}{8} \frac{2}{\frac{k\pi}{4}}\left(\sin\left(\frac{k\pi}{8}\right)\right) = \boxed{\frac{\sin\left(\frac{k\pi}{8}\right)}{\pi k}}$$

SO: Write $x(t) = \sum_k \frac{\sin\left(\frac{k\pi}{8}\right)}{\pi k} e^{+j\frac{\pi}{4}kt}$

$$y_1(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} u(t-\tau) \left(\sum_k e^{+j\frac{\pi}{4}k\tau} a_k\right) d\tau$$

$$= \int_{-\infty}^{\infty} \left\{\sum_k e^{+j\frac{\pi}{4}k\tau} a_k e^{-t} e^{\tau} u(t-\tau)\right\} d\tau$$

$$= \sum_k \int_{-\infty}^{\infty} e^{+j\frac{\pi}{4}k\tau} a_k e^{-t} e^{\tau} u(t-\tau)\, d\tau$$

$$= \sum_k a_k e^{-t} \int_{-\infty}^{t} e^{+j\frac{\pi}{4}k\tau} e^{\tau} d\tau = \sum_k a_k e^{-t} \left[\frac{e^{t(1+j\frac{\pi}{4}k)}}{(1+j\frac{\pi}{4}k)} - 0\right]$$

$$y_1(t) = \sum_k a_k \, e^{+j\frac{\pi}{4}kt} \left( \frac{1}{1+j\frac{\pi}{4}k} \right)$$

$$\therefore b_k = a_k \left( \frac{1}{1+j\frac{\pi}{4}k} \right) = \boxed{\frac{\sin\left(\frac{k\pi}{8}\right)}{\pi k} \left( \frac{1}{1+j\frac{\pi}{4}k} \right)}$$

$y_1(t)$ is periodic, $\quad y_1(t) = p(t) * \sum_n \delta(t-8n) = \sum_k b_k \, e^{+jk\omega_0 t}$

with fund. period $= \omega_0 = \frac{\pi}{4}$

The function $p(t)$ in this case is $\quad p(t) = \Pi(t) * e^{-t}u(t)$

$$\mathcal{F}\{p(t)\} = P(j\omega) = \frac{2\sin(\omega/2)}{\omega} \cdot \frac{1}{1+j\omega}$$

$$\mathcal{F}\{y_1(t)\} = \sum_k b_k \, 2\pi\delta(\omega - k\omega_0)$$ ← take FT of synthesis eq. for $y_1$

$$\mathcal{F}\{y_1(t)\} = P(j\omega) \cdot \frac{2\pi}{8} \sum_k \delta(\omega - k\omega_0) = \sum_k \frac{P(jk\omega_0)}{8} \, 2\pi\delta(\omega - k\omega_0)$$

$$\therefore b_k = \frac{P(jk\omega_0)}{8}$$

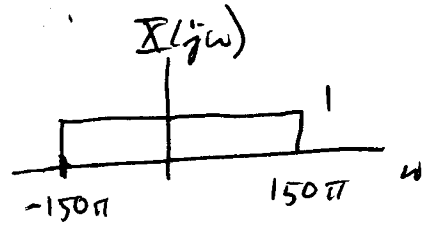
$$\frac{P(jk\omega_0)}{8} = \frac{2\sin\left(\frac{k\pi/4}{\cancel{2}}\right)}{8\,k\frac{\pi}{4}} \; \frac{1}{1+jk\frac{\pi}{4}} = \boxed{\frac{\sin\left(\frac{k\pi}{8}\right)}{k\pi} \; \frac{1}{1+jk\frac{\pi}{4}}}$$

This matches results of direct Fourier series analysis.

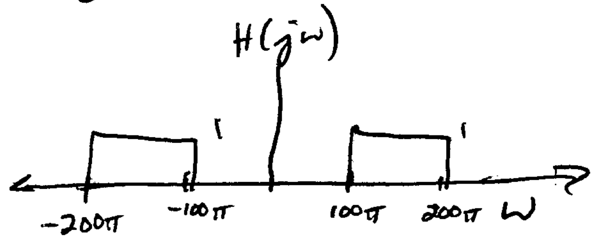
4) $x(t) = \dfrac{\sin(150\pi t)}{\pi t}$ $\qquad h(t) = \dfrac{\sin(200\pi t)}{\pi t} - \dfrac{\sin(100\pi t)}{\pi t}$

From table 4.2 (o.w.)

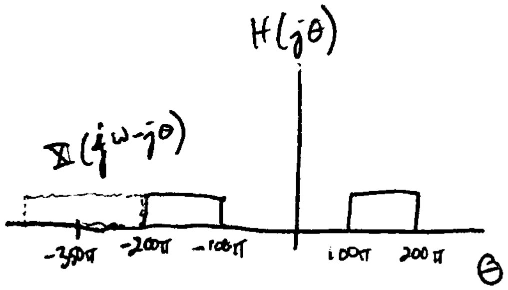
$$x(t) \Longleftrightarrow \begin{cases} 1 & |\omega| < 150\pi \\ 0 & \text{o.w.} \end{cases}$$

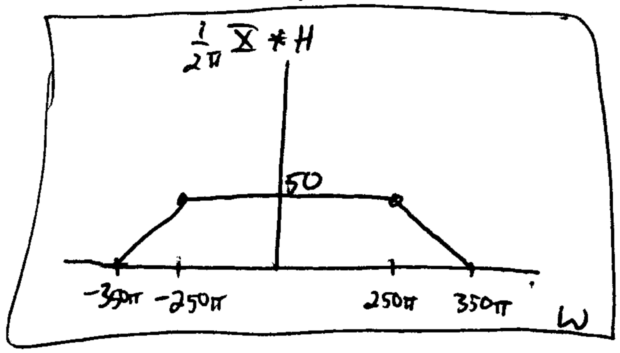



$$h(t) \Longleftrightarrow \left\{\begin{matrix} 1 & |\omega| < 200\pi \\ 0 & \text{o.w.} \end{matrix}\right\} - \left\{\begin{matrix} 1 & |\omega| < 100\pi \\ 0 & \text{o.w.} \end{matrix}\right\}$$




a) $x(t)\,h(t) \Longleftrightarrow \frac{1}{2\pi} X(j\omega) * H(j\omega)$

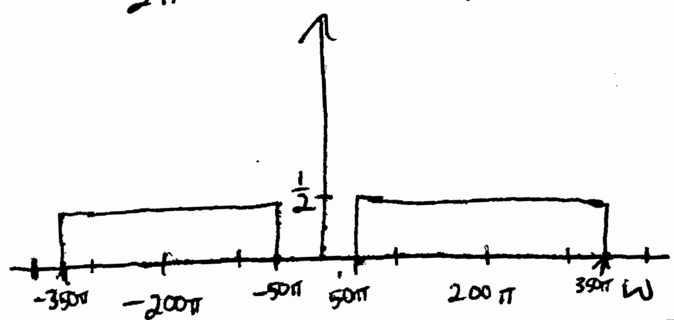


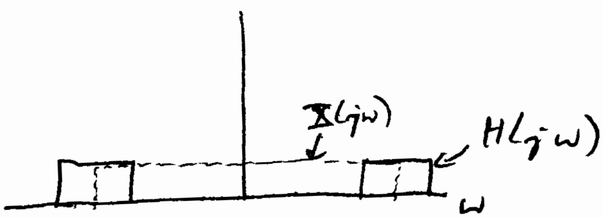
b) $f(t) = \cos(2\pi(100t)) \Longleftrightarrow \pi\left[\delta(\omega - 2\pi 100) + \delta(\omega + 2\pi 100)\right]$

Produces two copies of $X(j\omega)$, one @ $\omega = 200\pi$, another @ $\omega = -200\pi$

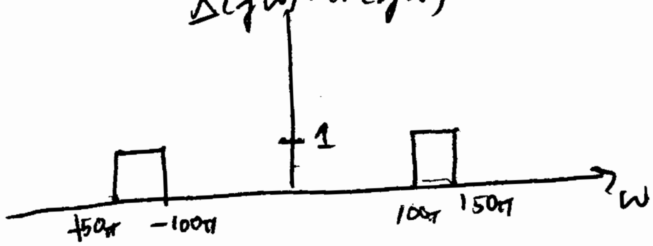
$\frac{1}{2\pi} X(j\omega) * F(j\omega)$




c) $x(t) * h(t) \Longleftrightarrow X(j\omega) \cdot H(j\omega)$

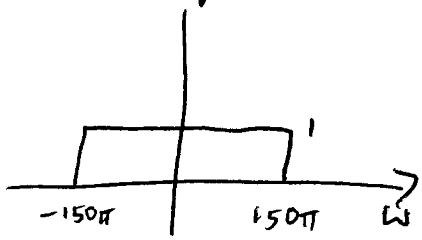



$X(j\omega) \cdot H(j\omega)$

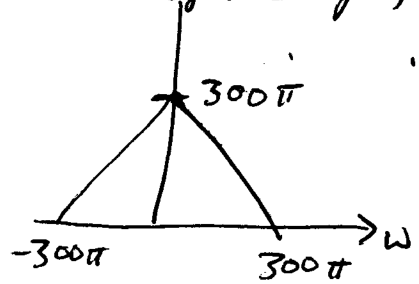


4d) $x^2(t) = x(t)\cdot x(t) \Longleftrightarrow \frac{1}{2\pi} X(j\omega) * X(j\omega)$

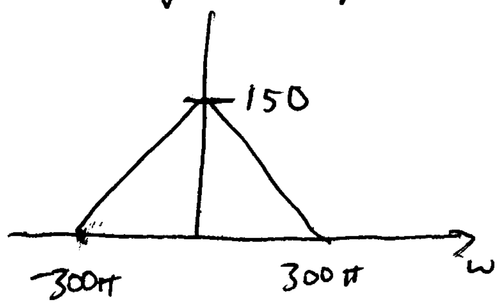
$X(j\omega)$




$X(j\omega) * X(j\omega)$

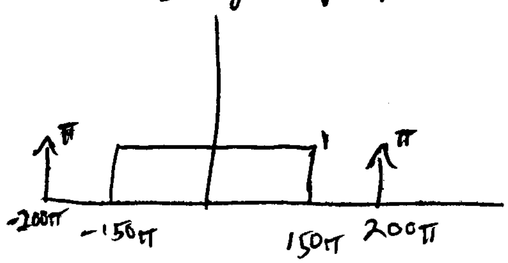



$\frac{1}{2\pi} X(j\omega) * X(j\omega)$

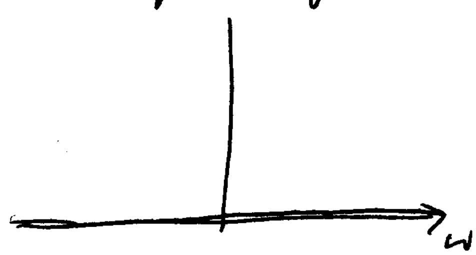



4e) $x(t) * \cos(2\pi(100)t) \Longleftrightarrow X(j\omega)\cdot\pi[\delta(\omega-200\pi) + \delta(\omega+200\pi)]$

$X(j\omega)$, $F(j\omega)$




$X(j\omega)\cdot F(j\omega) = 0$



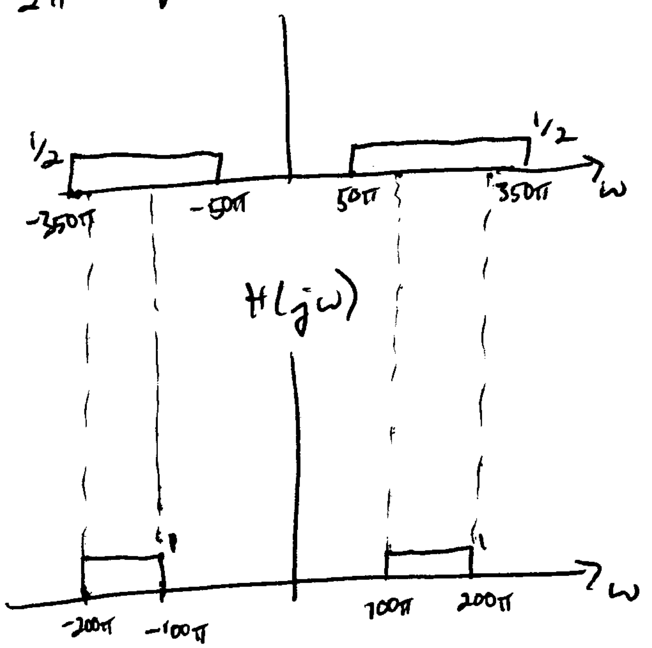
48) $[x(t)\cos(200\pi t)] * h(t)$

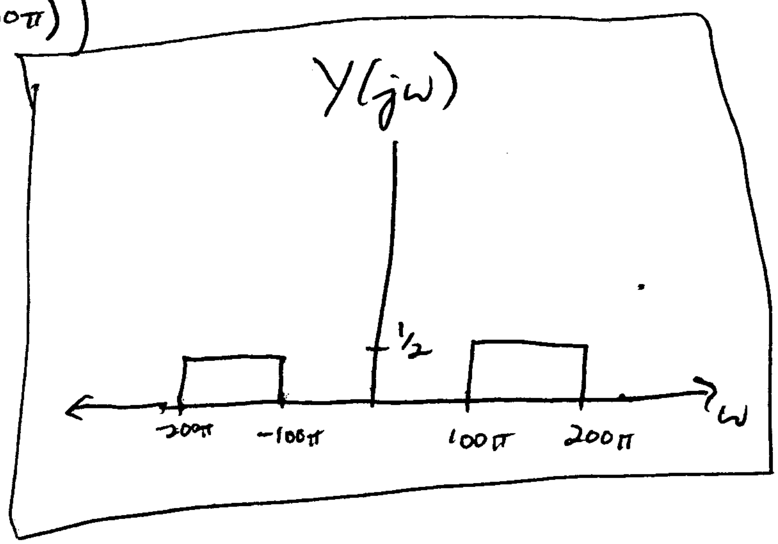
$$\Longleftrightarrow$$

$$\mathcal{F}\{x(t)\cos(200\pi t)\}\cdot H(j\omega).$$

$$=\left\{\frac{1}{2\pi} X(j\omega) * \pi[\delta(\omega-200\pi)+\delta(\omega+200\pi)]\right\}\cdot H(j\omega)$$

$\frac{1}{2\pi} X(j\omega) * \pi\{\delta(\omega-200\pi)+\delta(\omega+200\pi)\}$



5a) $x(t)$ can be written: $x(t) = \frac{x(t)}{2} + \frac{x(t)}{2} + \frac{x(-t)}{2} - \frac{x(-t)}{2}$

Let $x_1(t) = \frac{x(t)}{2} + \frac{x(-t)}{2}$, so $x_1(-t) = x_1(t)$

$\therefore$ $\boxed{x_1(t) \text{ is even } = x_e(t)}$

Let $x_2(t) = \frac{x(t)}{2} - \frac{x(-t)}{2}$, so. $x_2(-t) = -x_2(t)$

$\therefore$ $\boxed{x_2(t) \text{ is odd } = x_o(t)}$

---

5b): $x(t)$ is real, $(x(t) = x^*(t))$

Write even function: $x_e(t) = \frac{x(t)}{2} + \frac{x(-t)}{2}$

Write odd function: $x_o(t) = \frac{x(t)}{2} - \frac{x(-t)}{2}$

$X_e(j\omega) = \mathcal{F}\{x_e(t)\} = \frac{1}{2}\left[\mathcal{F}\{x(t)\} + \mathcal{F}\{x(-t)\}\right]$, $X_o(j\omega) = \frac{1}{2}\left[\mathcal{F}\{x(t)\} + -\mathcal{F}\{x(-t)\}\right]$

$\mathcal{F}\{x(-t)\} = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt = \int_{+\infty}^{-\infty} x(\tau)e^{+j\omega\tau}d\tau(-1) = \int_{-\infty}^{\infty} x(\tau)e^{+j\omega\tau}d\tau = X^*(j\omega)$

Let $\tau = -t$
$d\tau = -dt$

thus $X_e(j\omega) = \frac{1}{2}(X(j\omega) + X^*(j\omega)) = Re\{X(j\omega)\}$ □

$\mathcal{F}^*\{x(-t)\} = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = \mathcal{F}\{x(t)\}$, or, $X^*(j\omega) = \mathcal{F}\{x(-t)\}$

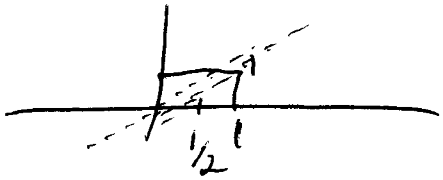
So: $X_e(j\omega) = \frac{1}{2}\left[X(j\omega) + X^*(j\omega)\right] = \frac{1}{2}\left[Re\{X\} + jIm\{X\} + Re\{X\} - jIm\{X\}\right]$

$$\boxed{X_e(j\omega) = Re\{X(j\omega)\}, \text{ QED}}$$

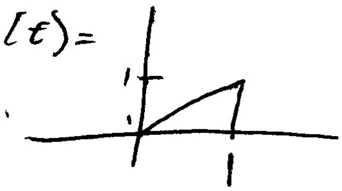
$X_o(j\omega) = \frac{1}{2}\left[X(j\omega) - X^*(j\omega)\right] = \frac{1}{2}\left[Re\{X\} + jIm\{X\} - Re\{X\} + jIm\{X\}\right]$

$$\boxed{X_o(j\omega) = jIm\{X(j\omega)\}, \text{ QED}}$$

5c) $x(t) = t\Pi(t-\frac{1}{2})$




$x(t) =$



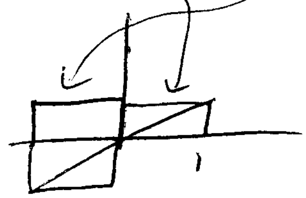

$$x_e(t) = \frac{x(t)}{2} + \frac{x(-t)}{2} = \frac{1}{2}\left(t\Pi(t-\tfrac{1}{2}) - t\Pi(-t-\tfrac{1}{2})\right)$$

$$x_e(t) = \frac{1}{2}t\left(\underbrace{\Pi(t-\tfrac{1}{2})}\, - \underbrace{\Pi(-t-\tfrac{1}{2})}\right)$$

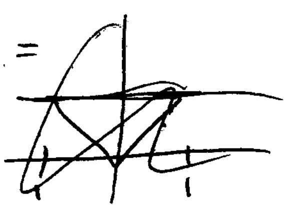
$-t-\frac{1}{2} = \frac{1}{2}$

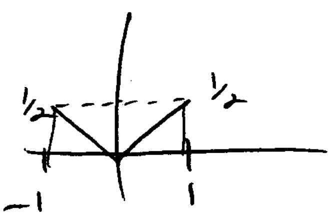
$-t = 1$

$t = -1$




$x_e(t) =$

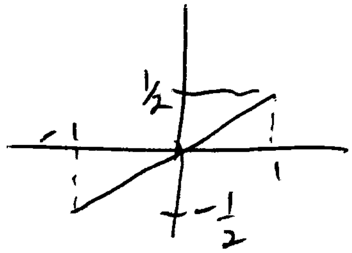





$$x_o(t) = \frac{x(t)}{2} - \frac{x(-t)}{2} = \frac{1}{2}(t)\left(\Pi(t-\tfrac{1}{2}) +^{+}(+\cancel{t})\,\Pi(-t-\tfrac{1}{2})\right)$$

$$x_o(t) = \frac{1}{2}t\left(\Pi(t-\tfrac{1}{2}) + \Pi(-t-\tfrac{1}{2})\right)$$

$x_o(t)$



$$\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\int_0^1 t e^{-j\omega t} dt = \frac{t e^{-j\omega t}}{-j\omega}\Bigg|_0^1 - \int_0^1 \frac{e^{-j\omega t}}{-j\omega} dt$$

$u = t \qquad dv = e^{-j\omega t} dt$

$du = dt \qquad v = \frac{e^{-j\omega t}}{-j\omega}$

$$\frac{e^{-j\omega}}{-j\omega} - \frac{e^{-j\omega t}}{(+j\omega)(+j\omega)}\Bigg|_0^1 = \frac{e^{-j\omega}}{j\omega} - \left(\frac{e^{-j\omega}}{-\omega^2} - e^0\right)$$

$$= \frac{e^{-j\omega}}{-j\omega} + \frac{e^{-j\omega} - 1}{\omega^2}$$

$$X(j\omega) = j\frac{e^{j\omega}}{\omega} + \frac{e^{-j\omega}}{\omega^2} - \frac{1}{\omega^2}$$

$$\boxed{X_e(j\omega) = -\frac{\sin(\omega)}{\omega} + \frac{\cos(\omega)}{\omega^2} - \frac{1}{\omega^2}}$$

$$\boxed{X_o(j\omega) = j\left(\frac{\cos(\omega)}{\omega} - \frac{\sin(\omega)}{\omega^2}\right)}$$