EE 120 PS#5 Fall 2016 Ming Jin

1a

CTFT property: $x(t) * y(t) \longleftrightarrow X(j\omega)Y(j\omega)$

We know that $h(t) \leftrightarrow H(j\omega)$

and $\sum_{n=-\infty}^{\infty} \delta(t-nT) \longleftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T} n)$

[why? First find the Fourier series:

$$a_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} \delta(t-nT) e^{-jk\frac{2\pi}{T} \cdot t} dt$$

$$= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\frac{2\pi}{T}t} dt$$ (all other impulses fall out of the range $[-\frac{T}{2}, \frac{T}{2}]$)

$$= \frac{1}{T}, \ \forall k,$$

therefore, we can write

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$ (synthesis eqn.).

taking the FT of the RHS, and $e^{jk\frac{2\pi}{T}t} \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega - k\frac{2\pi}{T})$

we have: $\sum_{n=-\infty}^{\infty} \delta(t-nT) \longleftrightarrow \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}k)$]

Therefore, for $x(t) = h(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$,

$$X(j\omega) = \mathcal{F}\{x(t)\} = H(j\omega)\, \mathcal{F}\{\sum_{n=-\infty}^{\infty} \delta(t-nT_0)\}$$

$$= H(j\omega) \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T_0}k)$$

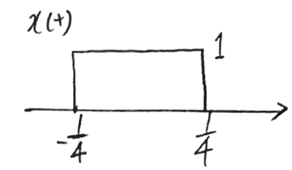
$$= \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_0} H(j\frac{2\pi}{T_0}k)\, \delta(\omega - \frac{2\pi}{T_0}k)$$

Since $\mathcal{F}\{x(t)\} = \mathcal{F}\{\sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T_0}t}\} = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\frac{2\pi}{T_0})$

$\Rightarrow a_k = \frac{1}{T_0} H(j\frac{2\pi}{T_0}k)$ by comparison.

1b

$$\Pi(2t) * \mathrm{comb}(\tfrac{t}{5}) = y(t).$$



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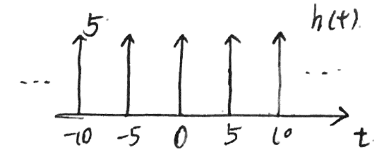


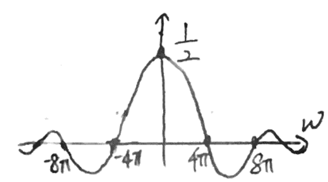
$$\Pi(2t) = u(2t+\tfrac{1}{2}) - u(2t - \tfrac{1}{2})$$
$$= u(2(t+\tfrac{1}{4})) - u(2(t-\tfrac{1}{4}))$$
$$= \begin{cases} 1 & -\frac{1}{4} \le u \le \frac{1}{4} \\ 0 & \text{o.w.} \end{cases}$$

$$\mathrm{comb}(\tfrac{t}{5}) = \sum_{n=-\infty}^{\infty} \delta(\tfrac{t}{5} - n)$$
$$= \sum_{n=-\infty}^{\infty} \delta(\tfrac{1}{5}(t-5n))$$
$$= 5 \sum_{n=-\infty}^{\infty} \delta(t-5n)$$

$$X(j\omega) = \frac{\sin \omega/4}{\omega/2}$$

$$H(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{5})$$

$\Downarrow$



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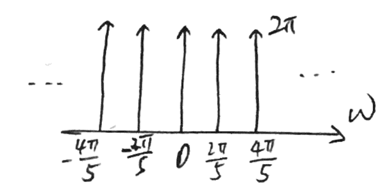
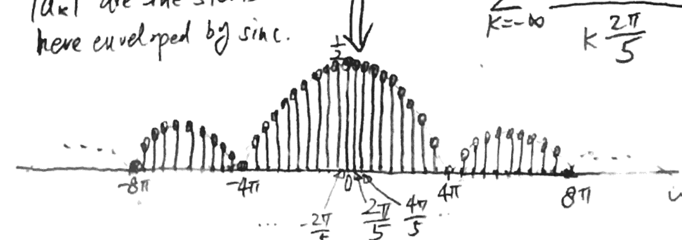


$$Y(j\omega) = X(j\omega) H(j\omega)$$
$$= \sum_{k=-\infty}^{\infty} \frac{4\pi \sin k\frac{\pi}{10}}{k\frac{2\pi}{5}} \delta(\omega - k\frac{2\pi}{5})$$

$|a_k|$ are the stems here enveloped by sinc.

$\Downarrow$



$$\omega_0 = \frac{2\pi}{5}. \quad a_k = \frac{2\sin k\frac{\pi}{10}}{k\frac{2\pi}{5}}$$

1c

Let $y(t) = \Pi(2t) * \text{comb}(\frac{t}{5})$ as in part (b)

$z(t) = y(t) * \delta(t-1)$ for the problem

$$Z(j\omega) = Y(j\omega)\, \mathcal{F}\{\delta(t-1)\} = Y(j\omega)\, e^{-j\omega}$$

$$= \sum_{k=-\infty}^{\infty} \frac{4\pi \sin k\frac{\pi}{10}}{k\frac{2\pi}{5}} e^{-jk\frac{2\pi}{5}} \delta(\omega - k\tfrac{2\pi}{5})$$

Since $T_0 = 5$, $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{5}$

and $a_k = \dfrac{2 \sin k\frac{\pi}{10}}{k\frac{2\pi}{5}} e^{-jk\frac{2\pi}{5}}$

Since $|a_k| = \left|\dfrac{2\sin k\frac{\pi}{10}}{k\frac{2\pi}{5}}\right| \left|e^{-jk\frac{2\pi}{5}}\right| = \left|\dfrac{2\sin k\frac{\pi}{10}}{k\frac{2\pi}{5}}\right|$,

the line spectrum is the same as (b)

1d Again let $y(t) = \Pi(2t) * \text{comb}(\frac{t}{5})$

$z(t) = y(t) * \Pi(2t)$

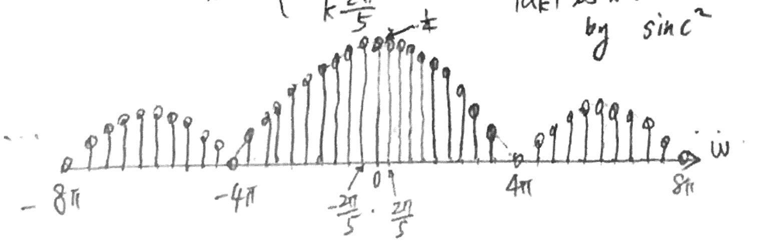
$$Z(j\omega) = Y(j\omega)\, \mathcal{F}\{\Pi(2t)\} = Y(j\omega) \frac{2\sin\frac{\omega}{4}}{\omega}$$

$$= \sum_{k=-\infty}^{\infty} 2\pi \left(\frac{2\sin k\frac{\pi}{10}}{k\frac{2\pi}{5}}\right)^2 \delta(\omega - k\tfrac{2\pi}{5})$$

As $T_0 = 5$, $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{5}$, we also have

$$a_k = \left(\frac{2\sin k\frac{\pi}{10}}{k\frac{2\pi}{5}}\right)^2$$

$|a_k|$ is the stem enveloped by sinc^2



1e Let $y(t) = \Pi(2t) * \text{comb}(\frac{t}{5})$

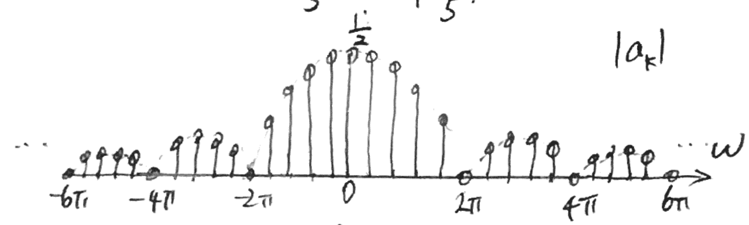
$z(t) = y(t) * \Pi(t)$

$$Z(j\omega) = Y(j\omega)\, \mathcal{F}\{\Pi(t)\} = Y(j\omega) \frac{2\sin\frac{\omega}{2}}{\omega}$$

$$= \sum_{k=-\infty}^{\infty} \frac{4\pi \sin k\frac{\pi}{10}}{k\frac{2\pi}{5}} \cdot \frac{2\sin k\frac{\pi}{5}}{k\frac{2\pi}{5}} \delta(\omega - k\tfrac{2\pi}{5})$$

As $T_0 = 5$, $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{5}$,

$$a_k = \frac{2\sin k\frac{\pi}{10}}{k\frac{2\pi}{5}} \cdot \frac{2\sin k\frac{\pi}{5}}{k\frac{2\pi}{5}}$$



Note: instead of zero-crossing at 4π multiples, they are at 2π multiples due to the term $\sin k\frac{\pi}{5}$.

② Using the transform pair $\delta(t-t_0) \leftrightarrow e^{-j\omega t_0}$

$$H(j\omega) = \mathcal{F}\left\{\sum_{k=0}^{\infty} e^{-kT}\delta(t-kT)\right\} = \sum_{k=0}^{\infty} e^{-kT}\mathcal{F}\{\delta(t-kT)\}$$

$$= \sum_{k=0}^{\infty} e^{-kT} e^{-j\omega kT} = \frac{1}{1-e^{-T-j\omega T}} \quad \text{(geometric sum)}$$

Therefore, $G(j\omega) = \frac{1}{H(j\omega)} = 1-e^{-T-j\omega T}$

$\boxed{\text{Time}}$ $x(t) \longrightarrow \boxed{h(t)} \longrightarrow \boxed{g(t)} \longrightarrow$ output

$x(t) \leftrightarrow X(j\omega)$ $\boxed{\text{Freq}}$

$x(t)*h(t) \leftrightarrow X(j\omega)H(j\omega)$

$x(t)*(h(t)*g(t)) = x(t) \leftrightarrow X(j\omega)(H(j\omega)G(j\omega)) = X(j\omega)$

Note: since $\delta(t) \leftrightarrow 1$, $\delta(t-T) \leftrightarrow e^{-j\omega T}$

$$g(t) = \delta(t) - e^{-T}\delta(t-T) \leftrightarrow G(j\omega) = 1 - e^{-T-j\omega T}$$

If we denote the received signal as $y(t)$, then to get back the original signal, we simply have

$$\hat{x}(t) = y(t) * g(t) = y(t) - e^{-T} y(t-T) = x(t)$$

Here $y(t) = x(t) * h(t) = \sum_{k=0}^{\infty} e^{-kT} x(t-kT)$.

To see that $g(t) * h(t) = \delta(t)$, lets derive as follow:

$$g(t) * h(t) = [\delta(t) - e^{-T}\delta(t-T)] * \sum_{k=0}^{\infty} e^{-kT}\delta(t-kT)$$

$$= \sum_{k=0}^{\infty} e^{-kT}\delta(t-kT) - \sum_{k=0}^{\infty} e^{-(k+1)T}\delta(t-kT-T)$$

$$= \sum_{k=0}^{\infty} e^{-kT}\delta(t-kT) - \sum_{k=1}^{\infty} e^{-kT}\delta(t-kT)$$

$$= \delta(t) \quad \text{Q.E.D.}$$

③a

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \quad \text{(analysis equ.)}$$

$$\mathcal{F}\{X(t)\} = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\,dt$$

Note: σ is a dummy var. not to confuse with t

$\boxed{\text{substitute in } \omega = t}$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\sigma)e^{-jt\sigma}\,d\sigma\right] e^{-j\omega t}\,dt$$

$\boxed{\text{exchange integral order}}$
$$= \int_{-\infty}^{\infty} x(\sigma)\left[\int_{-\infty}^{\infty} e^{-jt\sigma-j\omega t}\,dt\right] d\sigma$$

Since $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$,

i.e. $\int_{-\infty}^{\infty} e^{-j\omega_0 t} e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-j\omega_0 t - j\omega t}\,dt = 2\pi\delta(\omega+\omega_0)$

Let $\omega_0 = \sigma$, in the above we have $\int_{-\infty}^{\infty} e^{-jt\sigma-j\omega t}\,dt = 2\pi\delta(\omega+\sigma)$

Therefore, $\mathcal{F}\{X(t)\} = \int_{-\infty}^{\infty} x(\sigma)(2\pi\delta(\omega+\sigma))\,d\sigma$

$$= 2\pi x(-\omega)$$

by the sifting property

(3b) Since we know the Fourier transform pair

$$\cos(W_0 t) \longleftrightarrow \pi[\delta(w-W_0)+\delta(w+W_0)]$$

Therefore we have

$$X_1(jw) = \delta(w-2\pi)+\delta(w+2\pi)$$

$$= \mathcal{F}\{\frac{1}{\pi}\cos 2\pi t\} = \mathcal{F}\{x_1(t)\}$$

$$x_1(t) = \frac{1}{\pi}\cos 2\pi t$$

By the duality property,

$$\mathcal{F}\{X_1(t)\} = \mathcal{F}\{\delta(t-2\pi)+\delta(t+2\pi)\} = 2\pi x_1(-w) = 2\cos 2\pi w$$

(3c) Since $\Pi(\frac{t}{T}) = \begin{cases} 1 & |t| \le \frac{T}{2} \\ 0 & |t| > \frac{T}{2} \end{cases} \longleftrightarrow \frac{2\sin wT/2}{w}$

Therefore we have

$$X_2(jw) = \frac{\sin Ww}{\pi w} = \mathcal{F}\{\frac{1}{2\pi}\Pi(\frac{t}{2W})\} = \mathcal{F}\{x_2(t)\}$$

$$x_2(t) = \frac{1}{2\pi}\Pi(\frac{t}{2W})$$

By duality, $\mathcal{F}\{X_2(t)\} = \mathcal{F}\{\frac{\sin Wt}{\pi t}\} = 2\pi x_2(-w)$

$$= \Pi(\frac{w}{2W})$$

(3d) From the table, $\tilde{x}_3(t) = e^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \tilde{X}_3(jw) = \frac{1}{2+jw}$

By the duality, $\mathcal{F}\{\frac{1}{2+jt}\} = 2\pi \tilde{x}_3(-w)$

(<u>time reversal</u>) and $\mathcal{F}\{\frac{1}{2-jt}\} = 2\pi\tilde{x}_3(w) = 2\pi e^{-2w}u(w)$

Therefore, we know that

$$\mathcal{F}^{-1}\{X_3(jw)\} = \mathcal{F}^{-1}\{e^{-2w}u(w)\} = \frac{1}{2\pi}\frac{1}{2-jt}$$

$$= x_3(t)$$

By duality, $\mathcal{F}\{X_3(t)\} = 2\pi x_3(-w) = \frac{1}{2+jw}$

(3e) From the table, $\tilde{x}_4(t) = e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \tilde{X}_4(jw) = \frac{2}{1+w^2}$

By the duality, $\mathcal{F}\{\frac{2}{1+t^2}\} = 2\pi\tilde{x}_4(-w)$

and $\mathcal{F}\{\frac{2}{1+t^2}\} = 2\pi\tilde{x}_4(w)$

Therefore, we know that

$$\mathcal{F}^{-1}\{X_4(jw)\} = \mathcal{F}^{-1}\{e^{-|w|}\} = \frac{1}{2\pi}\frac{2}{1+t^2}$$

$$= x_4(t)$$

By duality, $\mathcal{F}\{X_4(t)\} = 2\pi x_4(-w)$

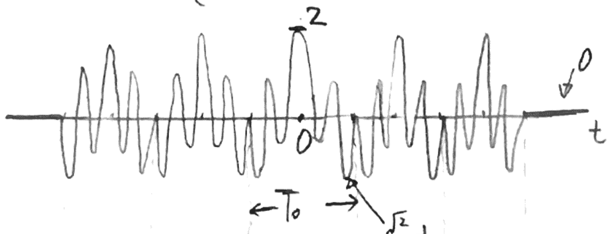
$$= \frac{2}{1+w^2}$$

Note: for (d) an (e), the idea is to first find $x(t)$, i.e., the IFT of the given $X(jw)$, and then apply the duality property to find $\mathcal{F}\{X(t)\}$.

4a

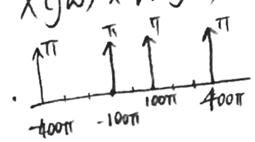
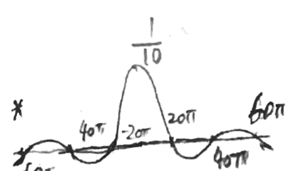
$z(t) = x(t)w(t)$

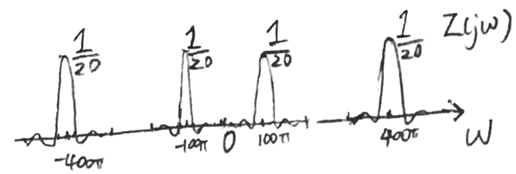
$= \left(\cos(100\pi t) + \cos(400\pi t)\right)\Pi(10t)$



$T_0 = \frac{1}{50}$

$Z(j\omega) = \frac{1}{2\pi}\; X(j\omega) * W(j\omega)$

$= \frac{1}{2\pi}\cdot$  $*$ 

$=$ 

$y(t) = z(t) * h(t) = z(t) * \delta(t) = z(t)$

$Y(j\omega) = Z(j\omega)$ (same as above)

$\boxed{\text{Note}}$ $X(j\omega) = \pi\left[\delta(\omega+100\pi) + \delta(\omega-100\pi) + \delta(\omega+400\pi) + \delta(\omega-400\pi)\right]$

$W(j\omega) = \dfrac{2\sin\frac{\omega}{20}}{\omega}$

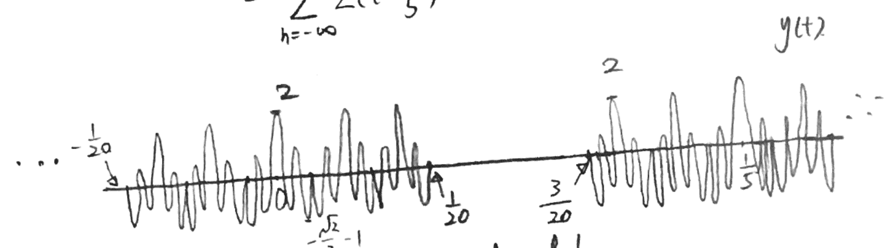
4b

Since $w(t)$ is the same, $z(t)$ is the same as 4a, and so is $Z(j\omega)$.

$y(t) = z(t) * h(t)$

$= z(t) * \left(\sum_{n=-\infty}^{\infty} \delta(t - \frac{n}{5})\right)$

$= \sum_{n=-\infty}^{\infty} z(t - \frac{n}{5})$

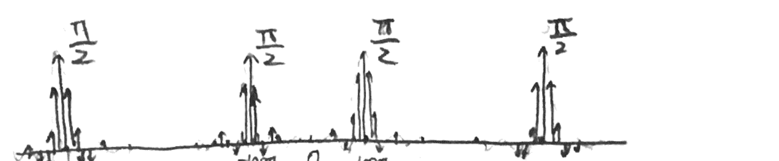


copies of $z(t)$ at multiples of $\frac{1}{5}$.

$Y(j\omega) = Z(j\omega)H(j\omega)$

$= Z(j\omega)\left(10\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 10\pi n)\right)$

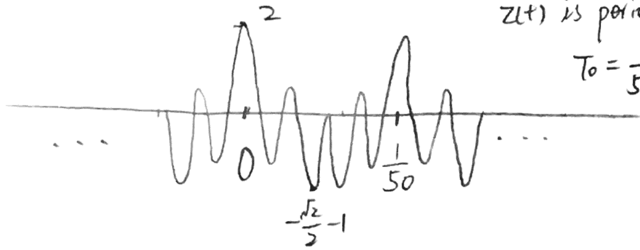
$= 10\pi \sum_{n=-\infty}^{\infty} Z(j10\pi n)\,\delta(\omega - 10\pi n)$



Samples of the envelope function from 4a at intervals 10π

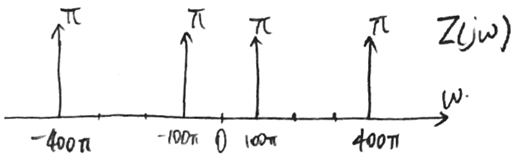
4c

$z(t) = x(t)\,w(t) = x(t).$



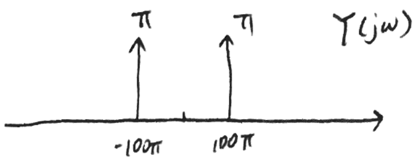
z(t) is periodic w/ period $T_0 = \frac{1}{50}$

$Z(j\omega) = X(j\omega) = \pi\left(\delta(\omega+100\pi) + \delta(\omega-100\pi) + \delta(\omega+400\pi) + \delta(\omega-400\pi)\right)$

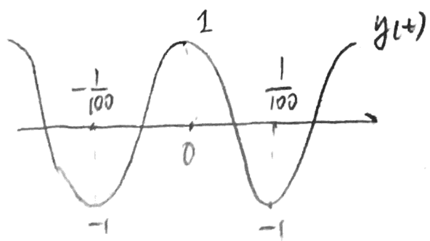


$H(j\omega) = \mathcal{F}\left\{\frac{\sin 200\pi t}{\pi t}\right\} = \Pi\left(\frac{\omega}{400\pi}\right)$ (by 3c)

Since $Y(j\omega) = Z(j\omega)H(j\omega)$,



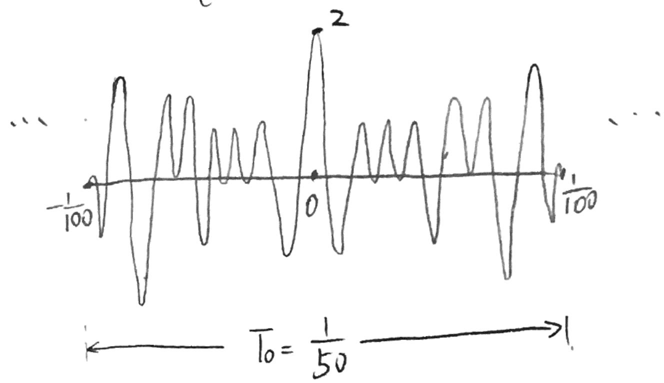
and $y(t) = \cos(100\pi t)$ from $Y(j\omega)$



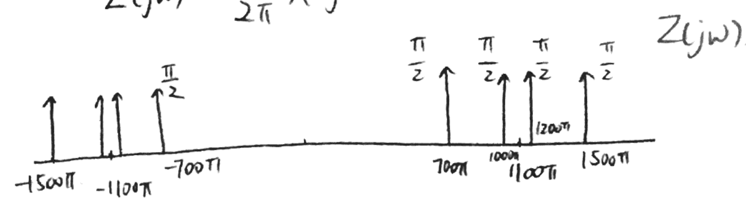
4d

$z(t) = x(t)\,w(t)$

$= \left(\cos(100\pi t) + \cos(400\pi t)\right)\cos 1100\pi t$

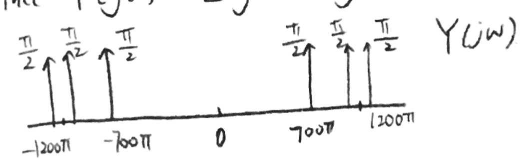


$Z(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$

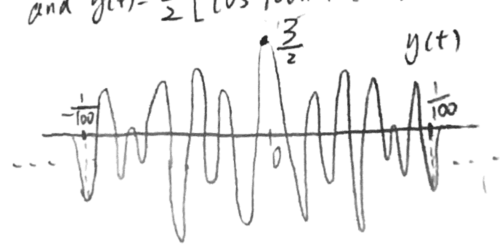


$H(j\omega) = \mathcal{F}\left\{\frac{\sin 1200\pi t}{\pi t}\right\} = \Pi\left(\frac{t}{2400\pi}\right)$

Since $Y(j\omega) = Z(j\omega)H(j\omega)$



and $y(t) = \frac{1}{2}\left[\cos 700\pi + \cos 1000\pi + \cos 1200\pi\right]$



$T_0 = \frac{2\pi}{100} = \frac{1}{50}$

6/7

⑤ (a)

$$x[n] = \delta[n+2] + 2\delta[n+1] + 2\delta[n-1] + \delta[n-2]$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$= e^{j2w} + 2e^{jw} + 2e^{-jw} + e^{-j2w}$$

$$= 4\cos w + 2\cos 2w$$

(b) $x[n] = \sin\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{8}n\right)$

$$= \frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j} + \frac{e^{j\frac{\pi}{8}n} + e^{-j\frac{\pi}{8}n}}{2}$$

Since $e^{jw_0 n} \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi\delta(w - w_0 - 2\pi k)$

$$X(e^{jw}) = \sum_{k=-\infty}^{\infty} \Big[\frac{1}{2j} 2\pi \delta(w - \frac{\pi}{4} - 2\pi k) - \frac{1}{2j} 2\pi \delta(w + \frac{\pi}{4} - 2\pi k) + \frac{1}{2} 2\pi \delta(w - \frac{\pi}{8} - 2\pi k) + \frac{1}{2} 2\pi \delta(w + \frac{\pi}{8} - 2\pi k) \Big]$$

Clearly $X(e^{jw})$ is periodic, for period $0 \le w < 2\pi$,

$$X(e^{jw}) = \pi\left[-j\delta(w-\frac{\pi}{4}) + j\delta(w+\frac{\pi}{4}) + \delta(w-\frac{\pi}{8}) + \delta(w+\frac{\pi}{8})\right]$$

(c). $H(e^{jw}) = \cos^2 w = \frac{1}{2}(\cos 2w + 1)$

Since $\delta[n] \xleftrightarrow{DTFT} 1$ and $\delta[n-n_0] \xleftrightarrow{DTFT} e^{-jwn_0}$

$$\cos 2w = \frac{1}{2}e^{j2w} + \frac{1}{2}e^{-j2w}$$

$$\frac{1}{2}\delta[n+2] + \frac{1}{2}\delta[n-2] \longleftrightarrow \frac{1}{2}e^{j2w} + \frac{1}{2}e^{-j2w} = \cos 2w$$

Therefore, $h[n] = \frac{1}{4}\delta[n+2] + \frac{1}{4}\delta[n-2] + \frac{1}{2}\delta[n]$

(d). $h[n] = \frac{1}{2\pi}\int_0^{2\pi} H(e^{jw}) e^{jwn} dw$

$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{8}} e^{jwn} dw + \frac{1}{2\pi}\int_{\frac{7\pi}{8}}^{2\pi} e^{jwn} dw$$

$$= \frac{1}{2\pi} \frac{e^{jwn}}{jn}\Big|_0^{\frac{\pi}{8}} + \frac{1}{2\pi}\frac{e^{jwn}}{jn}\Big|_{\frac{7\pi}{8}}^{2\pi}$$

$$= \frac{1}{j2\pi n}\Big[e^{j\frac{\pi}{8}n} \underbrace{-1 + e^{j2\pi n}}_{\text{cancels}} - e^{j\frac{7\pi}{8}n}\Big]$$

$$= \frac{1}{j2\pi n}\Big[e^{j\frac{\pi}{8}n} - e^{j\frac{7\pi}{8}n}\Big] = -\frac{e^{j\frac{\pi}{2}n}\sin\frac{3\pi}{8}n}{\pi n}$$

(e). $H(e^{jw}) = e^{-2jw}\cos(w) = e^{-2jw}\left[\frac{1}{2}e^{jw} + \frac{1}{2}e^{-jw}\right]$

$$= \frac{1}{2}e^{-jw} + \frac{1}{2}e^{-3jw}$$

Since $\delta[n-n_0] \xleftrightarrow{DTFT} e^{-jwn_0}$

$$h[n] = \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n-3]$$