EE120 Fall 2016: PS6 Solutions

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Problem 1.

Given: $\mathcal{F}\{x[n]\} = X(e^{j\omega})$

a) $\mathcal{F}\{Im\{x[n]\}\} = \mathcal{F}\left\{\frac{x[n]-x^*[n]}{2j}\right\}$

$$\mathcal{F}\{Im\{x[n]\}\} = \frac{\mathcal{F}\{x[n]\}}{2j} - \frac{\mathcal{F}\{x^*[n]\}}{2j} = \frac{1}{2j}X(e^{j\omega}) - \frac{1}{2j}\sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n}$$

$$\mathcal{F}\{Im\{x[n]\}\} = \frac{1}{2j}X(e^{j\omega}) - \frac{1}{2j}\left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^*$$

$$\boxed{\mathcal{F}\{Im\{x[n]\}\} = \frac{1}{2j}X(e^{j\omega}) - \frac{1}{2j}X^*(e^{-j\omega})}$$

b) $\mathcal{F}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}$

Using the substitution: $n' = -n$,

$$\mathcal{F}\{x[-n]\} = \sum_{n'=-\infty}^{\infty} x[n']e^{j\omega n'}$$

$$\boxed{\mathcal{F}\{x[-n]\} = X(e^{-j\omega})}$$

c) $\mathcal{F}\{Odd\{x[n]\}\} = \mathcal{F}\left\{\frac{x[n]}{2} - \frac{x[-n]}{2}\right\}$

$$\mathcal{F}\{Odd\{x[n]\}\} = \frac{1}{2}X(e^{j\omega}) - \frac{1}{2}\mathcal{F}\{x[-n]\}$$

$$\boxed{\mathcal{F}\{Odd\{x[n]\}\} = \frac{1}{2}X(e^{j\omega}) - \frac{1}{2}X(e^{-j\omega})}$$

Problem 2.

Given this LDE: $$y[n] + \frac{1}{2}y[n-1] = x[n]$$

a) Frequency Response?

$$Y(e^{j\omega}) + \frac{1}{2}Y(e^{j\omega})e^{-j\omega} = X(e^{j\omega})$$

$$Y(e^{j\omega})\left(1 + \frac{1}{2}e^{-j\omega}\right) = X(e^{j\omega})$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)}}$$

b) Output for $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$?

$$X_1(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$Y(e^{j\omega}) = X_1(e^{j\omega})\,H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)}\frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)}$$

When taking IDTFT, we have a formula of the form of $G(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)}$, with a frequency-scaling. First we take the inverse transform of $G(e^{j\omega})$ to get a discrete time signal $g[n]$ then apply the frequency scaling rule to get $y[n]$.

$$G(e^{j2\omega}) = \frac{1}{\left(1-\frac{1}{4}e^{-j[2\omega]}\right)}$$

$$G(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$g[n] = \left(\frac{1}{4}\right)^n u[n]$$

$\boxed{y[n] = \left(\frac{1}{4}\right)^{n/2} u[n/2]}$ for n even, and zero for n odd.

c) Output for $x_2[n] = \left(-\frac{1}{2}\right)^n u[n]$?

$$X_2(e^{j\omega}) = \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)}$$

$$Y(e^{j\omega}) = X_2(e^{j\omega})\,H(e^{j\omega}) = \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)}\frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)}$$

$$Y(e^{j\omega}) = \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)^2}$$

This has a form from the DTFT pair table, so we can find the output discrete time signal:

$$\boxed{y[n] = (n+1)\left(-\frac{1}{2}\right)^n u[n]}$$

d) Output for $x_3[n] = \delta[n] + \frac{1}{2}\delta[n-1]$?

$X_3(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}$

$Y(e^{j\omega}) = X_3(e^{j\omega})\,H(e^{j\omega}) = \left(1 + \frac{1}{2}e^{-j\omega}\right)\frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)} = 1$

A constant DTFT has a corresponding pair with a Kronecker delta function, so the output discrete time signal:

$\boxed{y[n] = \delta[n]}$

e) Output for $x_4[n] = \delta[n] - \frac{1}{2}\delta[n-1]$?

$X_4(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}$

$Y(e^{j\omega}) = X_3(e^{j\omega})\,H(e^{j\omega}) = \frac{\left(1-\frac{1}{2}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)} = \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)} - \frac{1}{2}\frac{e^{-j\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)}$

Using DTFT pairs and properties, we can directly write output discrete time signal and simplify:

$y[n] = \left(-\frac{1}{2}\right)^n u[n] - \frac{1}{2}\delta[n-1] * \left(\left(-\frac{1}{2}\right)^n u[n]\right)$

$y[n] = \left(-\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{2}\right)^n u[n-1]$

$\boxed{y[n] = \left(-\frac{1}{2}\right)^n (u[n] + u[n-1])}$

f) Output for $X_5(e^{j\omega}) = \frac{\left(1-\frac{1}{4}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)}$?

$Y(e^{j\omega}) = X_5(e^{j\omega})\,H(e^{j\omega}) = \frac{\left(1-\frac{1}{4}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)}\frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)}$

$Y(e^{j\omega}) = \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)^2} - \frac{1}{4}\frac{e^{-j\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)^2}$

Using DTFT pairs and properties, we can directly write output discrete time signal and simplify:

$y[n] = (n+1)\left(-\frac{1}{2}\right)^n u[n] - \frac{1}{4}\delta[n-1] * \left((n+1)\left(-\frac{1}{2}\right)^n u[n]\right)$

$y[n] = (n+1)\left(-\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left((n)\left(-\frac{1}{2}\right)^{n-1} u[n-1]\right)$

$\boxed{y[n] = \left(-\frac{1}{2}\right)^n \left\{(n+1)u[n] + \frac{1}{2}n \cdot u[n-1]\right\}}$

g) Output for $X_6(e^{j\omega}) = 1 + 2e^{-3j\omega}$?

$Y(e^{j\omega}) = X_5(e^{j\omega})\,H(e^{j\omega}) = \frac{\left(1+2e^{-3j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)}$

$Y(e^{j\omega}) = \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)} + 2\frac{e^{-3j\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)}$

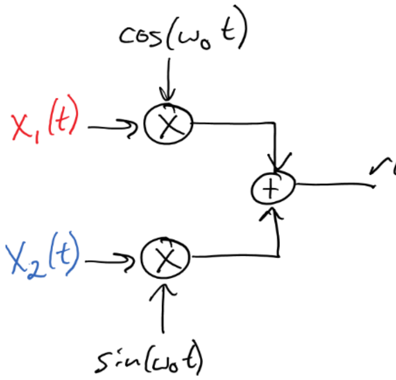
$y[n] = \left(-\frac{1}{2}\right)^n u[n] + 2\delta[n-3] * \left\{\left(-\frac{1}{2}\right)^n u[n]\right\}$

$y[n] = \left(-\frac{1}{2}\right)^n u[n] + 2\left(-\frac{1}{2}\right)^{n-3} u[n-3]$

$\boxed{y[n] = \left(-\frac{1}{2}\right)^n (u[n] - 16u[n-3])}$

Problem 3.

Part a.



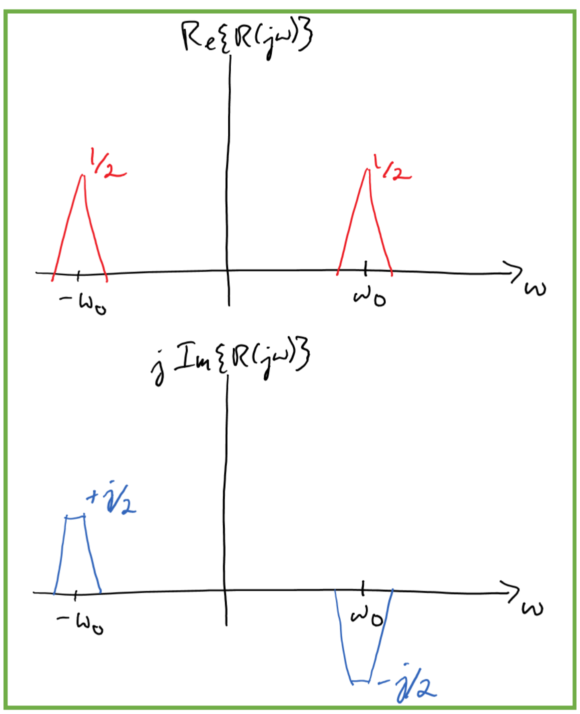
$r(t) = x_1(t)\cos(\omega_0 t) + x_2(t)\sin(\omega_0 t)$

$$R(j\omega) = \frac{1}{2\pi} X_1(j\omega) * \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$$+ \frac{1}{2\pi} X_2(j\omega) * \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$$

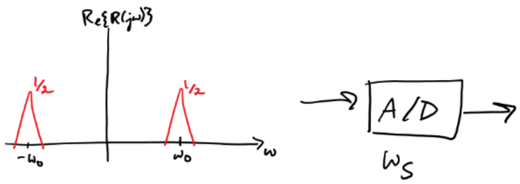
$X_1(j\omega)$ and $X_2(j\omega)$ are real, so $Re\{R(j\omega)\} = \frac{1}{2\pi} X_1(j\omega) * \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

$Im\{R(j\omega)\} = \frac{1}{2\pi} X_2(j\omega) * (-1)\pi[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

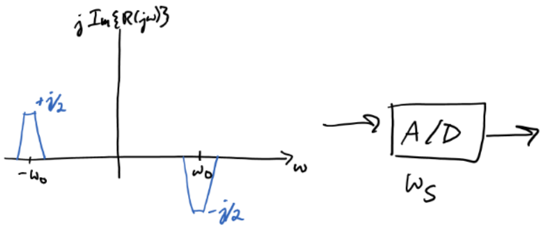


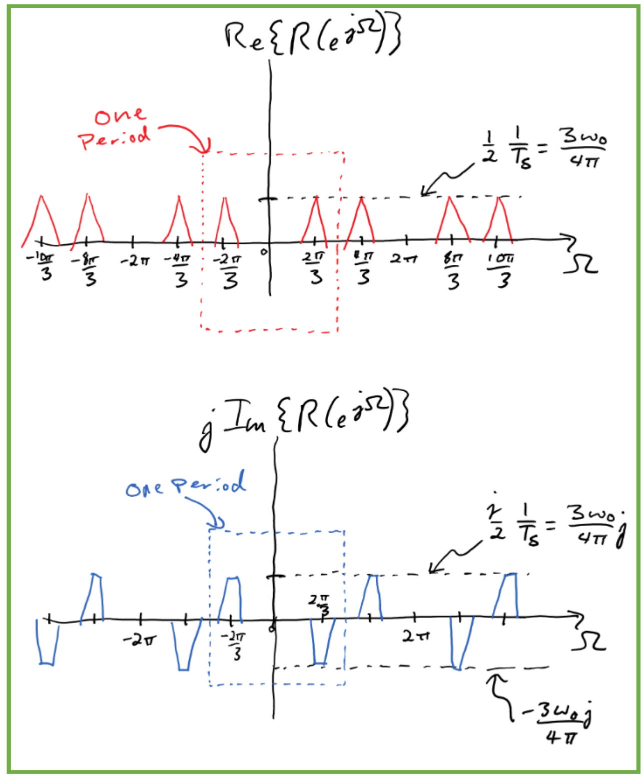
Part 3b.

3 b) r(t) is sampled.

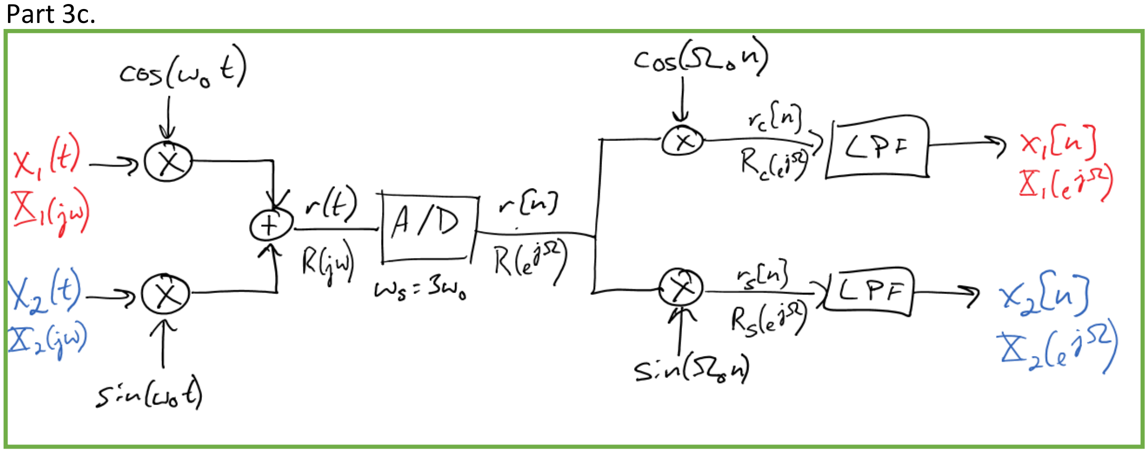


$$\Omega_0 = \frac{w_0}{w_s} 2\pi = \frac{2\pi}{3}$$





Part 3c.

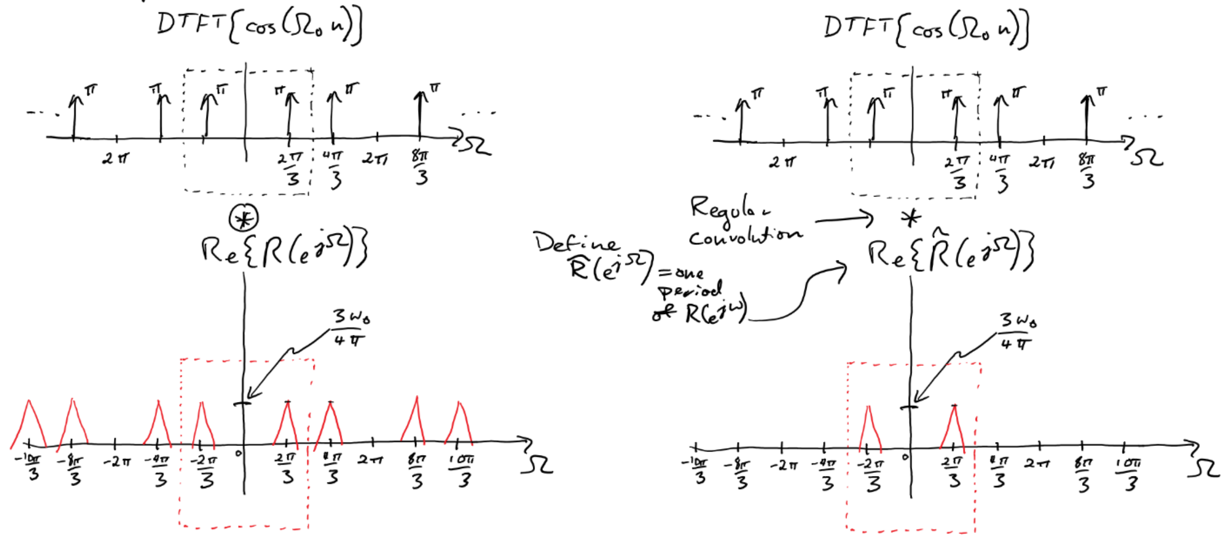


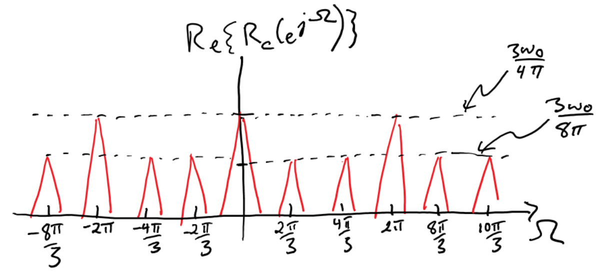
Part 3d.

$$r_c[n] = r[n]\cdot\cos\left(\frac{2\pi}{3}n\right) \xleftrightarrow{DTFT} \frac{1}{2\pi} R(e^{j\Omega}) \circledast \pi\sum_{\ell}\left[\delta\left(\Omega-\frac{2\pi}{3}-2\pi\ell\right)+\delta\left(\Omega+\frac{2\pi}{3}-2\pi\ell\right)\right] = R_c(e^{j\Omega})$$

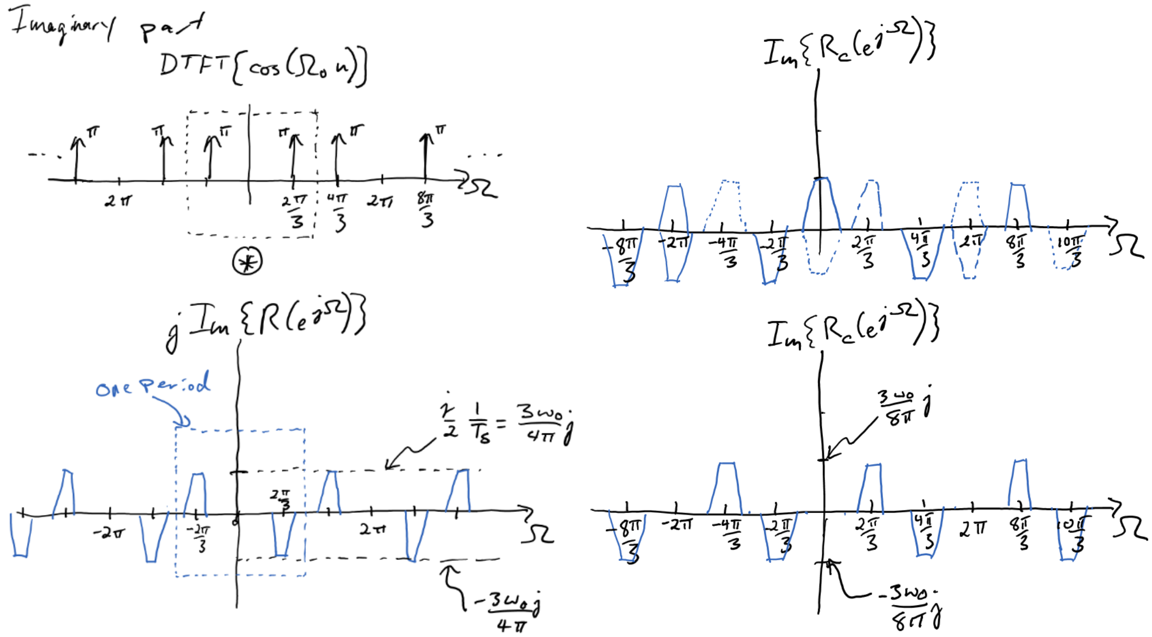
convolve both real and imaginary parts separately:

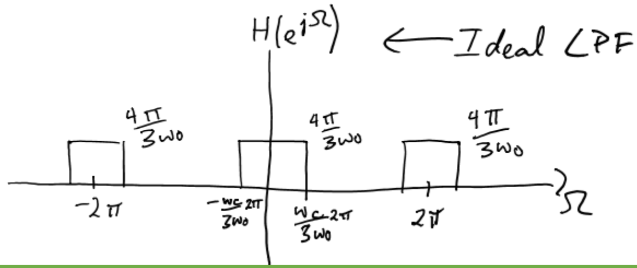
Real part

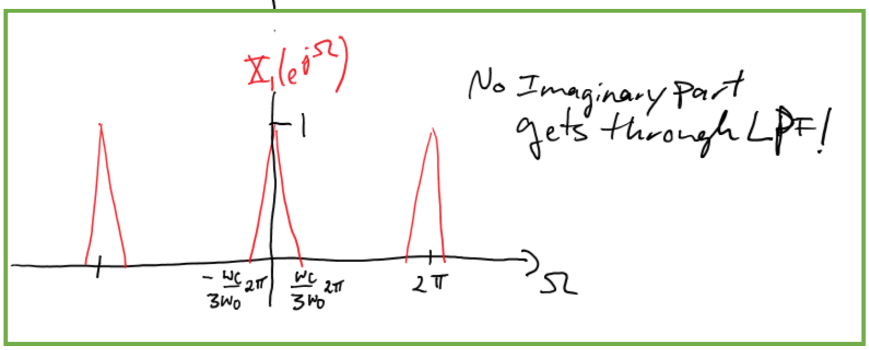




Imaginary part



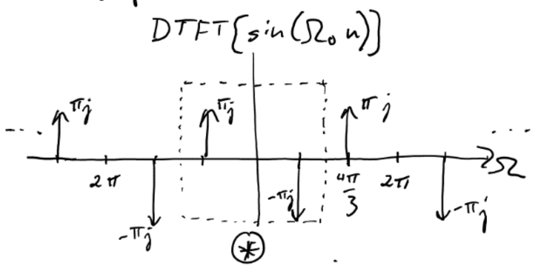


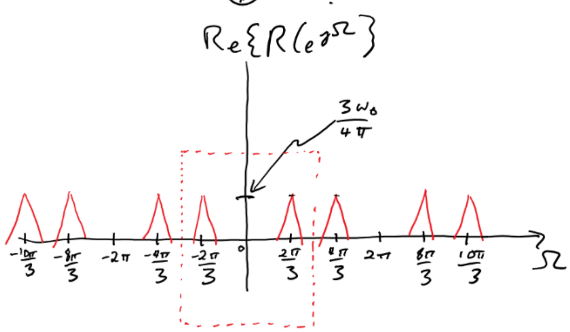


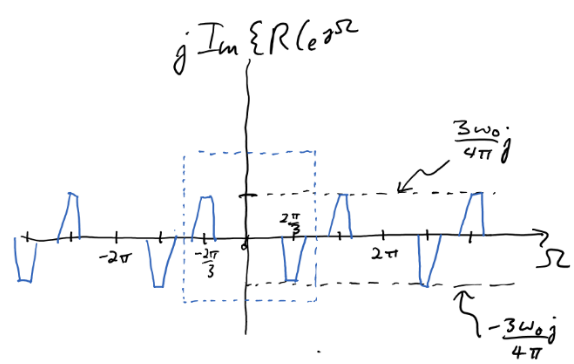
$$r_s[n] = r[n]\cdot\sin\left(\frac{2\pi}{3}n\right) \xleftrightarrow{DTFT} \frac{1}{2\pi} R(e^{j\Omega}) \circledast \frac{\pi}{j}\sum_{l}\left[\delta\left(\Omega-\frac{2\pi}{3}-2\pi l\right)-\delta\left(\Omega+\frac{2\pi}{3}-2\pi l\right)\right] = R_s(e^{j\Omega})$$

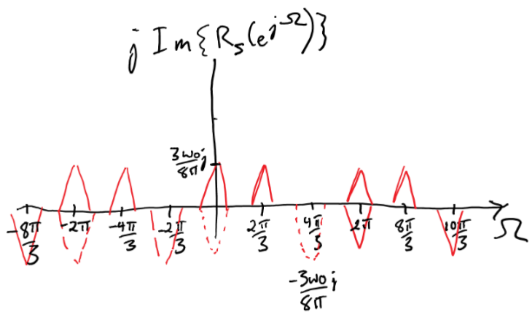
convolve both real and imaginary parts separately:

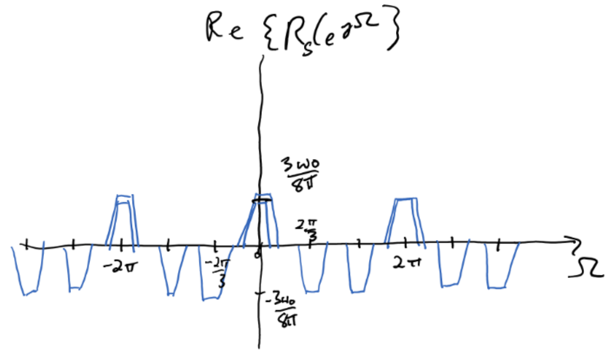
Real part

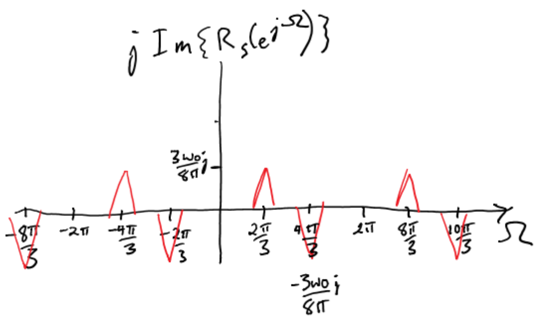


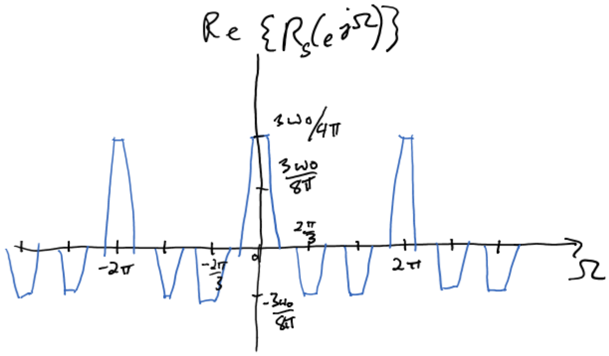


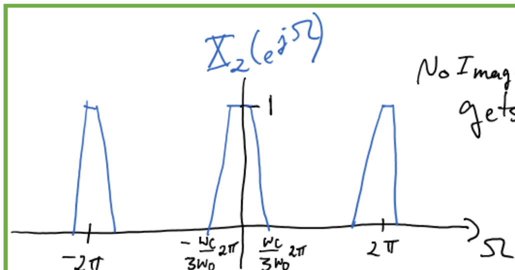












No Imaginary part of $R_s(e^{j\Omega})$ [Real part of $R(e^{j\Omega})$] gets through LPF!

Problem 4.

a) $H(e^{j\omega}) = e^{-j\omega}\frac{1-\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{-j\omega}}$

$$|H(e^{j\omega})|^2 = \left|e^{-j\omega}\frac{1-\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right|^2$$

$$|H(e^{j\omega})|^2 = \left|\frac{1-\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right|^2$$

$$|H(e^{j\omega})|^2 = \left[\frac{1-\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right]\cdot\left[\frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{j\omega}}\right] = 1$$

$\boxed{|H(e^{j\omega})| = 1}$

QED.

We have used the fact that for any complex number, z, $|z|^2 = zz^*$.

b) If we let:

$z_1 = e^{-j\omega} = r_1 e^{j\theta_1}$; $z_2 = 1 - \frac{1}{2}e^{j\omega} = r_2 e^{j\theta_2}$; and $z_3 = 1 - \frac{1}{2}e^{-j\omega} = r_3 e^{j\theta_3}$,

We can write $H(e^{j\omega})$:

$$H(e^{j\omega}) = \frac{z_1 z_2}{z_3} = \frac{r_1 r_2}{r_3}\exp(\theta_1 + \theta_2 - \theta_3)$$

We can write $\angle H(e^{j\omega}) = \theta_1 + \theta_2 - \theta_3$

$$\theta_1 = -\omega$$

$$z_2 = 1 - \frac{1}{2}e^{j\omega} = 1 - \frac{1}{2}\cos(\omega) - \frac{j}{2}\sin(\omega); \qquad \theta_2 = \text{atan}\left(-\frac{\sin(\omega)}{2}\frac{1}{1-\frac{1}{2}\cos(\omega)}\right)$$

$$z_3 = 1 - \frac{1}{2}e^{-j\omega} = 1 - \frac{1}{2}\cos(-\omega) - \frac{j}{2}\sin(-\omega); \qquad \theta_3 = \text{atan}\left(-\frac{\sin(-\omega)}{2}\frac{1}{1-\frac{1}{2}\cos(-\omega)}\right)$$

$$\theta_3 = \text{atan}\left(\frac{\sin(\omega)}{2}\frac{1}{1-\frac{1}{2}\cos(\omega)}\right)$$

Arctangent is an odd function, so $-\theta_2 = \theta_3$

$$\boxed{\angle H(e^{j\omega}) = \theta_1 + \theta_2 - \theta_3 = -\omega + 2\,\text{atan}\left(\frac{\sin(\omega)}{\cos(\omega)-2}\right)}$$

c) The input function, $x[n] = \cos\left(\frac{\pi}{3}n\right)$ can be expressed in basis functions:

$$x[n] = \frac{1}{2}e^{j\frac{\pi}{3}n} + \frac{1}{2}e^{-j\frac{\pi}{3}n}$$

Therefore, we can write the output of the system,

$$y[n] = \frac{1}{2}e^{j\frac{\pi}{3}n}\left|H\left(e^{j\frac{\pi}{3}}\right)\right|e^{j\angle H\left(e^{j\frac{\pi}{3}}\right)} + \frac{1}{2}e^{-j\frac{\pi}{3}n}\left|H\left(e^{-j\frac{\pi}{3}}\right)\right|e^{j\angle H\left(e^{-j\frac{\pi}{3}}\right)}$$

$$y[n] = \frac{1}{2}e^{j\frac{\pi}{3}n}e^{j\angle H\left(e^{j\frac{\pi}{3}}\right)} + \frac{1}{2}e^{-j\frac{\pi}{3}n}e^{j\angle H\left(e^{-j\frac{\pi}{3}}\right)}$$

$$\angle H\left(e^{j\frac{\pi}{3}}\right) = -\frac{\pi}{3} + 2\,\text{atan}\left(\frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)-2}\right) = -\frac{2\pi}{3}$$

$$\angle H\left(e^{-j\frac{\pi}{3}}\right) = \frac{\pi}{3} - 2\,\text{atan}\left(\frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)-2}\right) = -\angle H\left(e^{j\frac{\pi}{3}}\right) = \frac{2\pi}{3}$$

$$y[n] = \frac{1}{2}e^{j\frac{\pi}{3}n}e^{j2\pi/3} + \frac{1}{2}e^{-j\frac{\pi}{3}n}e^{-j2\pi/3}$$

$$y[n] = \frac{1}{2}e^{j\left(\frac{\pi}{3}n+\frac{2\pi}{3}\right)} + \frac{1}{2}e^{-j\left(\frac{\pi}{3}n+\frac{2\pi}{3}\right)}$$

$$\boxed{y[n] = \cos\left(\frac{\pi}{3}n + \frac{2\pi}{3}\right)}$$

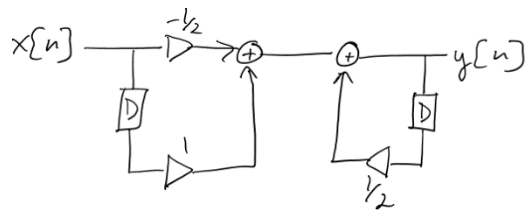
d)

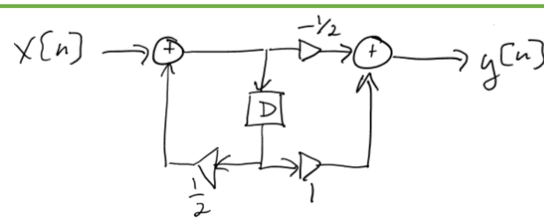
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$Y(e^{j\omega}) - \frac{1}{2}Y(e^{j\omega})e^{-j\omega} = X(e^{j\omega})e^{-j\omega} - \frac{1}{2}X(e^{j\omega})$$

$$\boxed{y[n] - \frac{1}{2}y[n-1] = x[n-1] - \frac{1}{2}x[n]}$$

$$y[n] = -\frac{1}{2}x[n] + x[n-1] + \frac{1}{2}y[n-1]$$





Problem 5.

a) $h(t) = e^{-2t}u(t)$

Take CTFT:

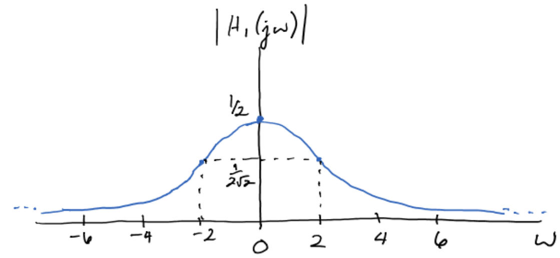
$\mathcal{F}\{h(t)\} = H_1(j\omega) = \frac{1}{2+j\omega}$

Key points on the magnitude graph:

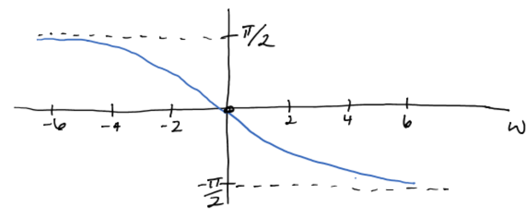
$|H_1(j0)| = 1/2$

$|H_2(\pm j2)| = \frac{1}{2\sqrt{2}}$ (3dB point)

$$|H_1(j\omega)| = \frac{1}{\sqrt{4+\omega^2}}$$



$$\measuredangle H_1(j\omega) = -\operatorname{atan}\left(\frac{\omega}{2}\right)$$



b) $h_2[n] = (1 - e^{-0.1})e^{-0.1n}u[n]$

Take DTFT:

$DTFT\{h_2[n]\} = H_2(e^{j\Omega}) = \frac{(1-e^{-0.1})}{1-e^{-0.1-j\Omega}}$

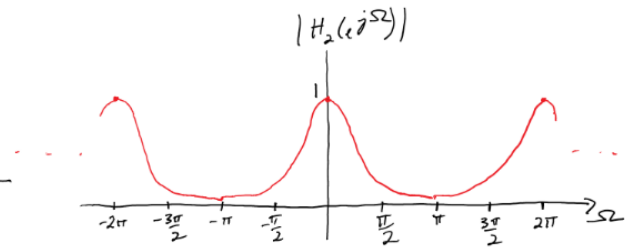
Key points on the magnitude graph:

$|H_2(e^{j0})| = 1$

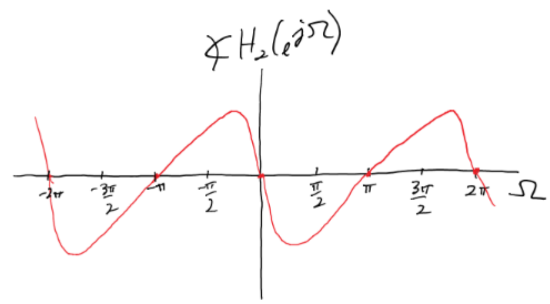
$|H_2(e^{j\pi})| \cong 0.05$

Periodic in $\Omega_0 = 2\pi$

$$|H_2(e^{j\Omega})| = \sqrt{1+a^2 - 2a\cos(\omega)}$$



$$\measuredangle H_2(e^{j\Omega}) = \operatorname{atan}\left(\frac{\sin(\omega)}{\cos(\omega) - \frac{1}{a}}\right)$$



c) We can make the filters match by selecting a sampling frequency, since $\Omega = \frac{\omega}{\omega_s} 2\pi$

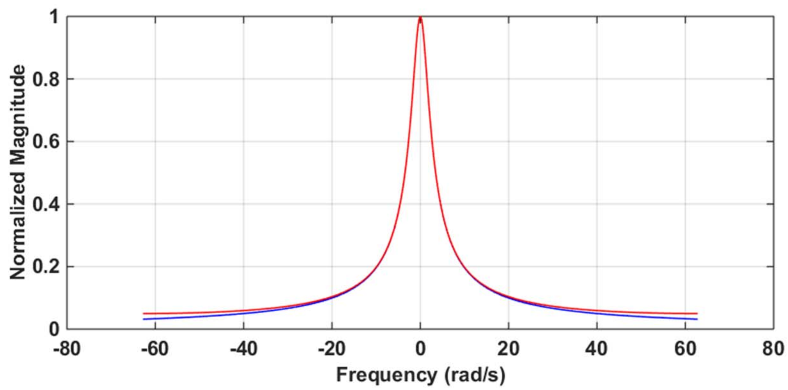
First, normalize $H'_1(j\omega) = 2H_1(j\omega)$

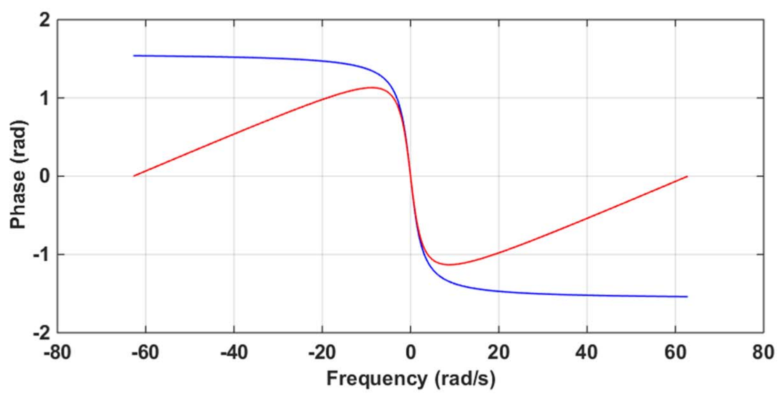
$\Omega_{3dB} \cong 0.1$ by inspection. We know $\Omega_{3dB} = \frac{\omega_{3dB}}{\omega_s} 2\pi$, and $\omega_{3dB} = 2$. We can solve for $\omega_s =$ 125.6 rad/s.

The discrete filter behaves like the continuous time filter inside one period of $H_2(e^{j\Omega})$. The period of H_2 in CTFT variable ω corresponds to

$$\Omega = \frac{\omega_{filter}}{\omega_s} 2\pi = \pi$$

So ω_{filter} is 63 rad/s, or ~10Hz.





However, it can be seen that the filter only has a monotonic phase response inside $\pm 3\pi$ rad/s, so it may be relevant to restrict the bandwidth further if the application called for no dispersion (linear phase).