EE120 Fall 2016

PS7 Solutions

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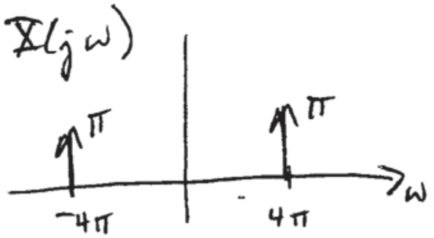
$\boxed{\#1}$ a) $x(t) = \cos(4\pi t) = \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right)$

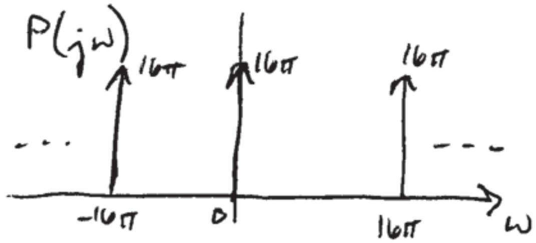
$$\tilde{x}(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{8}\right)$$

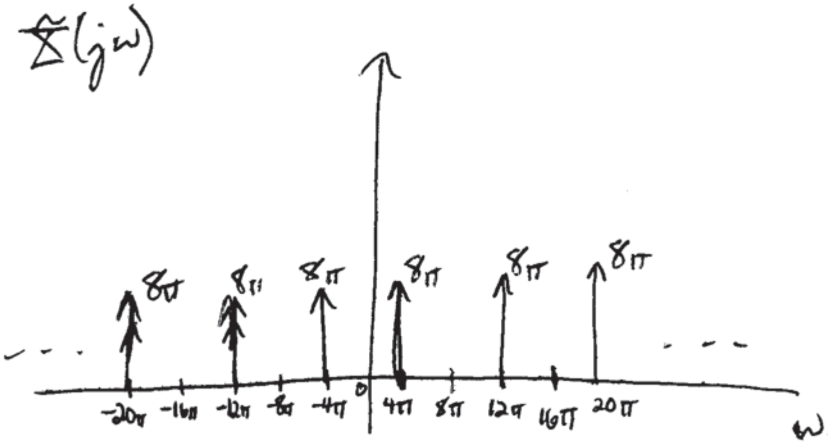
Define $P(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{8}\right)$

$$\tilde{X}(j\omega) = X(j\omega) * P(j\omega)\,\frac{1}{2\pi}$$

$$P(j\omega) = \frac{2\pi}{1/8} \sum_{k=-\infty}^{+\infty} \delta(\omega - 16\pi k)$$







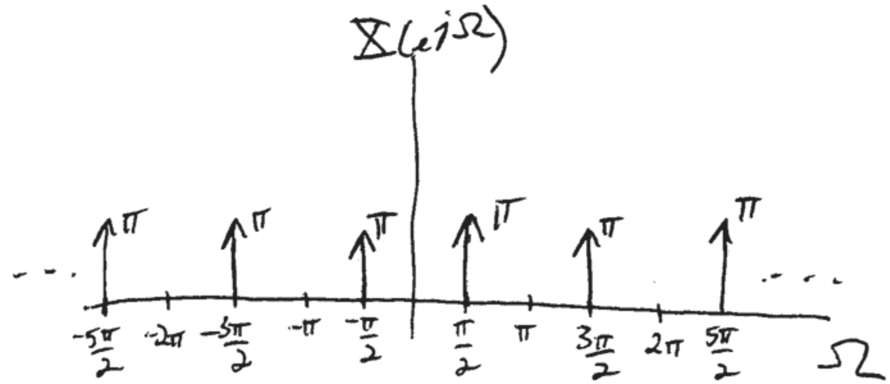
$\omega = 4\pi \rightarrow \Omega = \frac{\omega}{\omega_s} 2\pi = \frac{\pi}{2}$

$\omega = 16\pi \rightarrow \Omega = \frac{\omega}{\omega_s} 2\pi = 2\pi$

b) $x[n] = \cos\left(\frac{\pi}{2} n\right) = \frac{1}{2}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)$

$$e^{j\Omega_0 n} \xleftrightarrow{\text{DTFT}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi l)$$

$$X(e^{j\Omega}) = \frac{1}{\cancel{2}} \cancel{2}\pi \sum_{l=-\infty}^{\infty} \delta\left(\Omega - \frac{\pi}{2} - 2\pi l\right) + \delta\left(\Omega + \frac{\pi}{2} - 2\pi l\right)$$



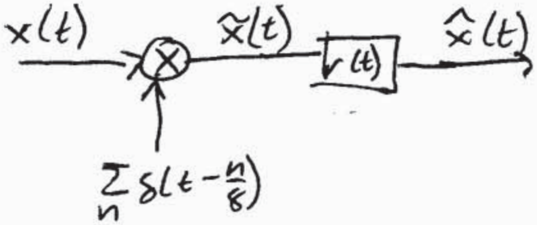
If we let $\Omega = \frac{\omega}{16\pi} \cdot 2\pi = \frac{\omega}{8}$

$\omega = 8\Omega$

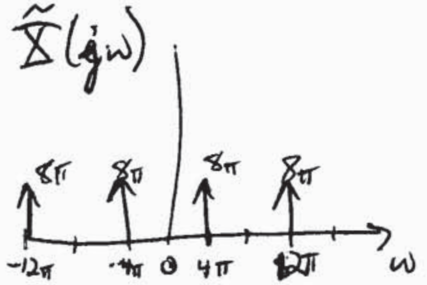
then $\boxed{X(e^{j\Omega}) = \frac{1}{8} \tilde{X}(j8\Omega)}$

Also:

c) We have

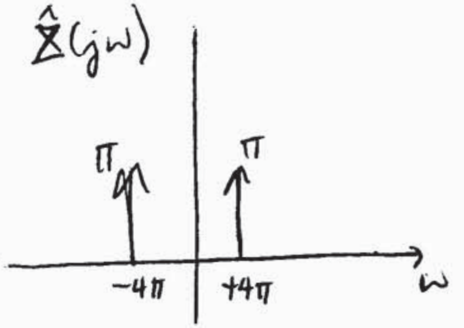


We want $r(t)$ s.t. $\hat{x}(t) = x(t)$



If we choose LPF so that we isolate the delta's @ $\pm 4\pi$, We will have $X(j\omega)$, scaled by 8

So: choose $R(j\omega)$



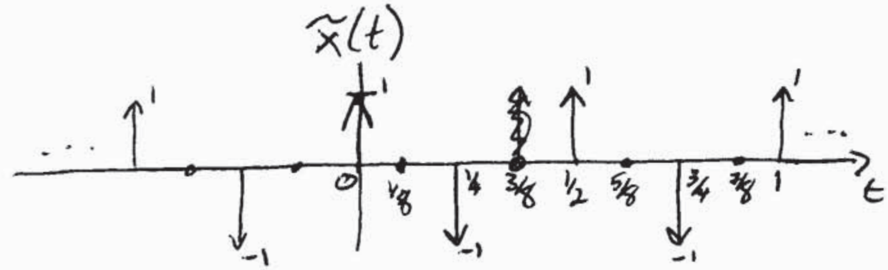
since $\hat{X}(j\omega) = X(j\omega)$,
$\underline{\hat{x}(t) = x(t)}$

1D) $\hat{x} = \tilde{x} * r$

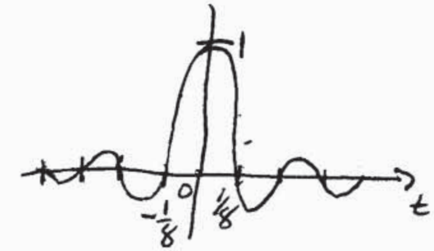
$$\tilde{x}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - \tfrac{n}{8}) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - \tfrac{n}{8})$$

$$\tilde{x}(t) = \sum_n \cos(nT_s 4\pi)\,\delta(t - \tfrac{n}{8})$$

$$\tilde{x}(t) = \sum_n \cos(\tfrac{\pi}{2} n)\,\delta(t - \tfrac{n}{8})$$



$$r(t) = \frac{1}{8}\,\frac{\sin(8\pi t)}{\pi t}$$



convolution of $\tilde{x}(t)$ with $r(t)$: place sincs, $r(t)$, @ each delta and add:

$$\hat{x}(t = \tfrac{1}{10}) = \text{area}(\tilde{x}(0))\,r(\tfrac{1}{10}) + \text{area}(\tilde{x}(\tfrac{1}{4}))\,r(\tfrac{1}{10} - \tfrac{1}{4}) + \cdots$$

$$\boxed{\hat{x}(t = \tfrac{1}{10}) = \sum_{n=-\infty}^{\infty} \text{area}(\tilde{x}(\tfrac{n}{8}))\,r(t - \tfrac{n}{8})}$$

(1d) cont'd:

$$\hat{x}\left(t=\tfrac{1}{16}\right) \approx \frac{1}{8}\frac{\sin\left(8\pi\frac{1}{16}\right)}{\pi\frac{1}{16}} - \frac{1}{8}\frac{\sin\left(8\pi\left(\frac{1}{16}-\frac{1}{4}\right)\right)}{\pi\left(\frac{1}{16}-\frac{1}{4}\right)}$$

$$- \frac{1}{8}\frac{\sin\left(8\pi\left(\frac{1}{16}+\frac{1}{4}\right)\right)}{\pi\left(\frac{1}{16}+\frac{1}{4}\right)} + \frac{\frac{1}{8}\sin\left(8\pi\left(\frac{1}{16}-\frac{1}{2}\right)\right)}{\pi\left(\frac{1}{16}-\frac{1}{2}\right)}$$

$$+ \frac{1}{8}\frac{\sin\left(8\pi\left(\frac{1}{16}+\frac{1}{2}\right)\right)}{\pi\left(\left(\frac{1}{16}\right)+\frac{1}{2}\right)} + \cdots$$

Truncated series

$$\boxed{\hat{x}\left(\tfrac{1}{16}\right) \approx 0.7013}$$

$$\hat{x}\left(\tfrac{1}{16}\right) = \cos\left(4\pi\tfrac{1}{16}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$\frac{\sqrt{2}}{2} \sim 0.707$, close

to our approximate value ✓

this sum simplifies:

$$\hat{x}\left(t=\tfrac{1}{16}\right) = \frac{1}{8\pi}\left(\frac{\sin\left(\frac{\pi}{2}\right)}{1/16} - \frac{\sin\left(\frac{\pi}{2}\right)}{-3/16} - \frac{\sin\left(\frac{\pi}{2}\right)}{5/16} + \frac{\sin\left(\pi/2\right)}{-7/16} + \frac{\sin\left(\pi/2\right)}{9/16} + \cdots\right.$$

$$\hat{x}\left(t=\tfrac{1}{16}\right) = \frac{2}{\pi}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \cdots\right)$$

Interesting to note: this sum can be

Written: $\frac{2}{\pi}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n(-2)}{(16n^2-1)}\right) = \hat{x}\left(\tfrac{1}{16}\right)$

evaluation out to 7 terms yields a result with less than 0.1% error!

#2 $x(t) = \cos(2\pi \cdot 500t)$

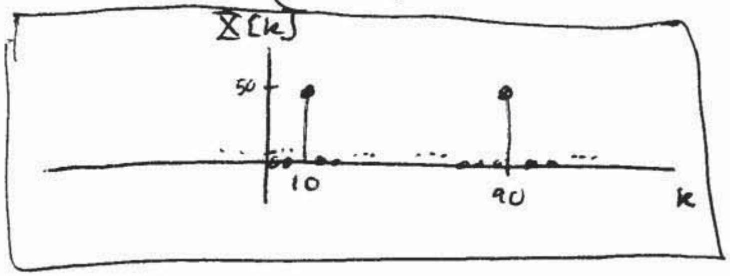
$$x[n] = x(nT_s) = \cos(1000\pi\,(200e\text{-}6)\,n)$$

$$x[n] = \cos\left(\frac{\pi}{5}n\right) = \frac{1}{2}\left(e^{j\frac{\pi}{5}n} + e^{-j\frac{\pi}{5}n}\right)$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{100}\sum_{k=0}^{99} X[k]\, e^{j\frac{2\pi kn}{100}}$$

$$X[k] = \begin{cases} 50 & k = 10, 90 \\ 0 & \text{o.w.} \end{cases}$$



frequency spacing is related to total time for original signal:

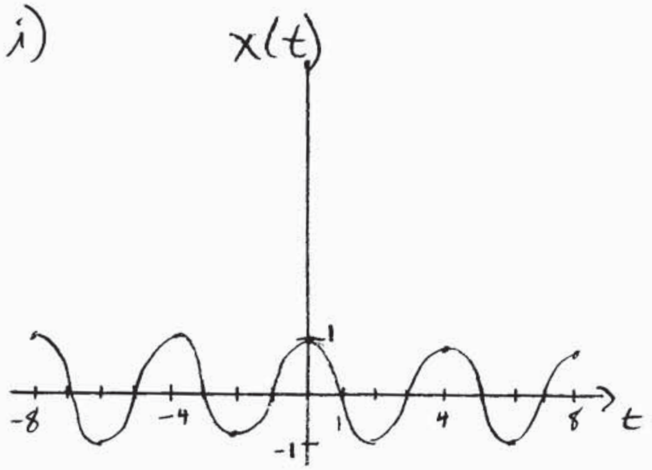
$$\Delta\omega = \frac{2\pi}{100T_s} = \frac{2\pi}{20\,ms} = \boxed{2\pi(50\ Hz)}$$

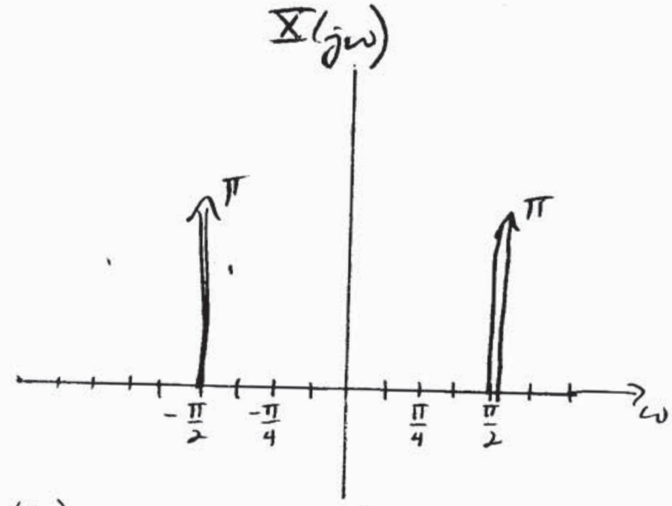
So: $k=10$ corresponds to $10 \cdot 50\,Hz = \boxed{500\,Hz}$ ($k=10$)

500 Hz corresponds to $(2\pi \cdot 500)$ rad/s

$$= \boxed{1000\pi\ \frac{rad}{s}}$$

3 i)

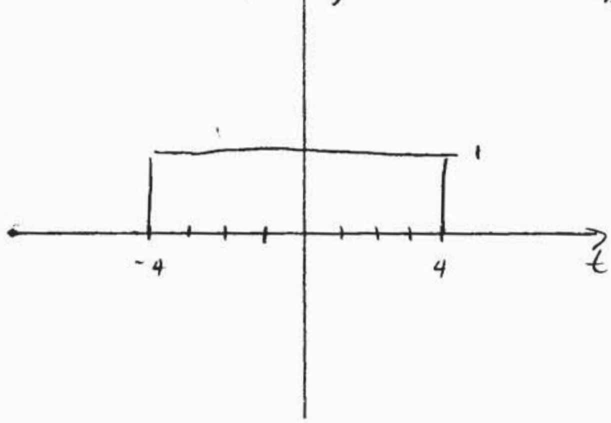


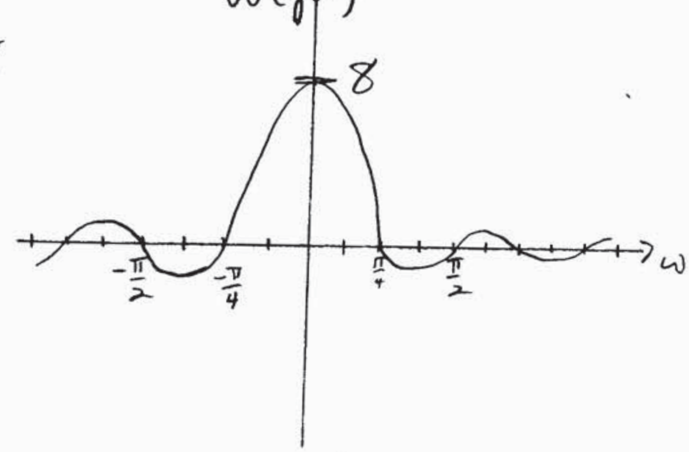


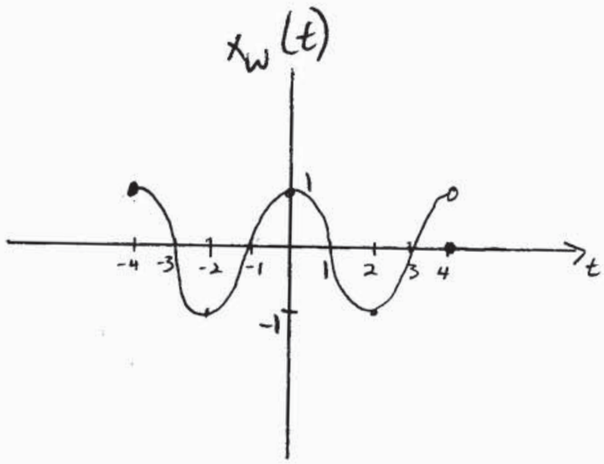
$W(t) = \Pi\left(\frac{2t}{T_0}\right) = \Pi\left(\frac{t}{4}\right) \rightarrow \frac{2\sin(4\omega)}{\omega} = W(j\omega)$

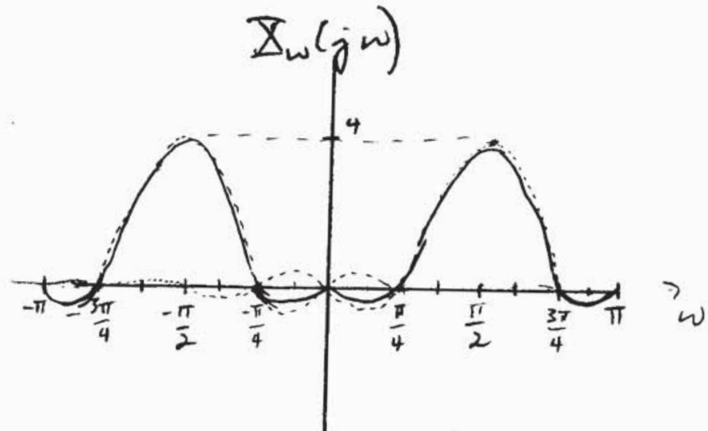
$4\omega = n\pi$

$\omega = n\frac{\pi}{4}$



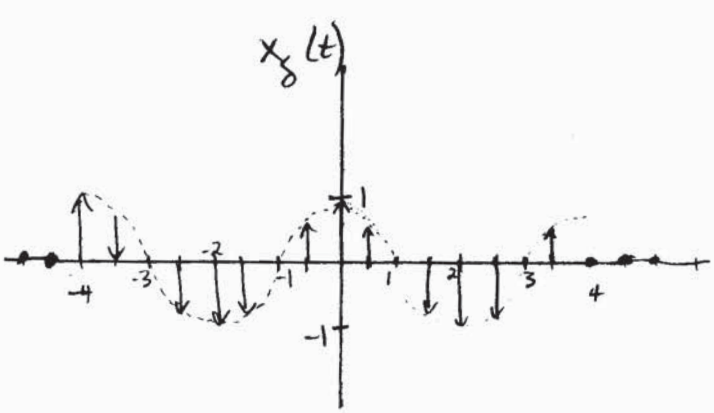


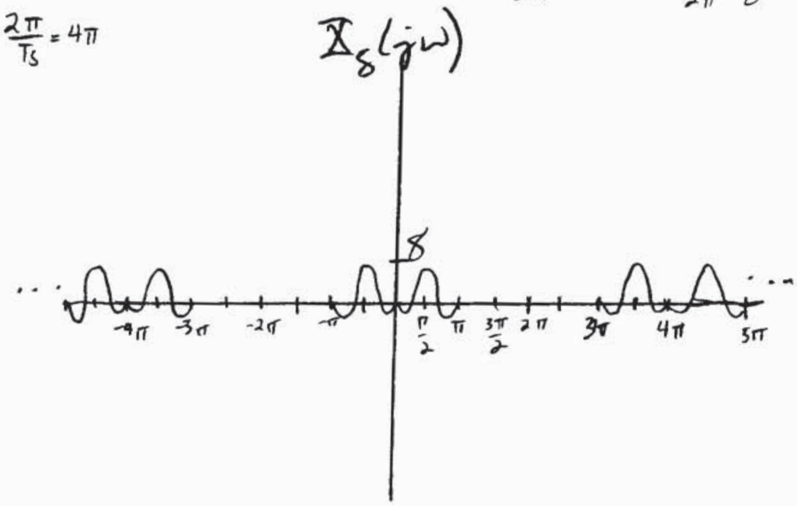




$\frac{1}{2\pi} 4 \cdot 4\pi = \frac{16\pi}{2\pi} = 8$

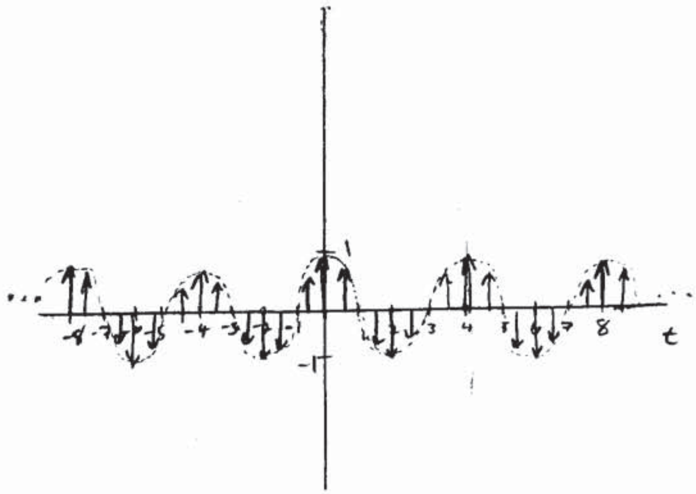
$\frac{2\pi}{T_s} = 4\pi$





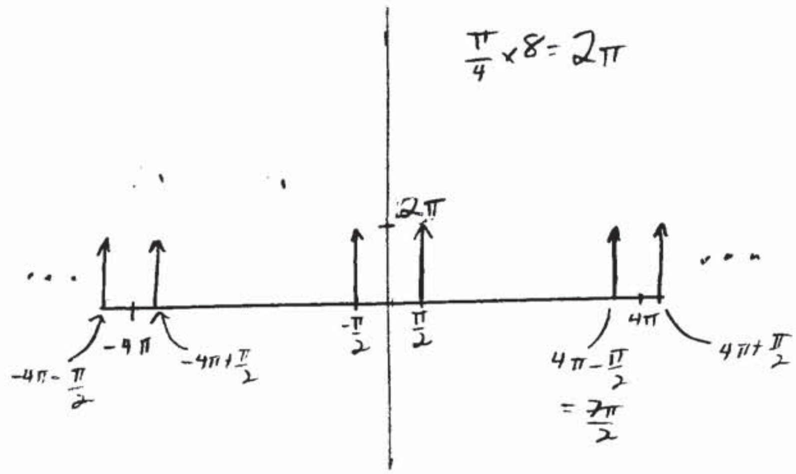
3 i)

$x'(t)$

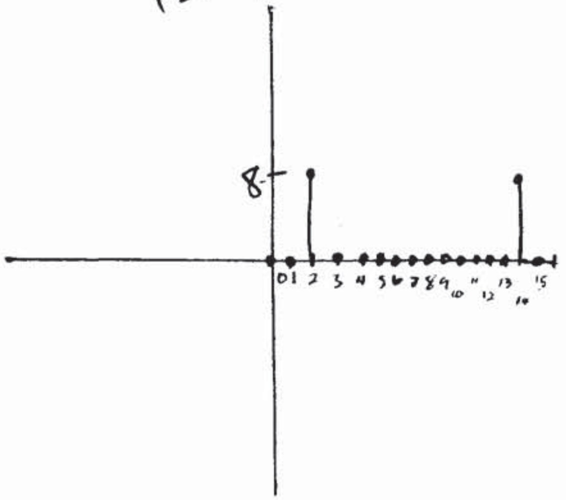


$X'(j\omega)$

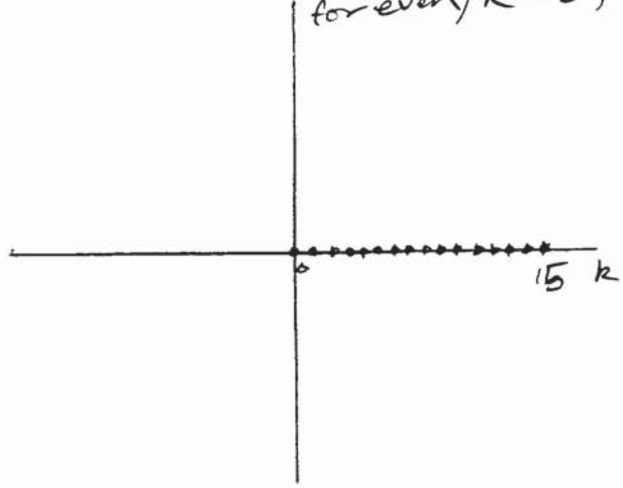
$\frac{\pi}{4} \times 8 = 2\pi$



$|X[k]|$



$\measuredangle X[k] = 0$ for every $k \in [0, 15]$



$$X[k] = \frac{T_0}{2\pi} \, Area\left(X'\left(jk\frac{2\pi}{T_0}\right)\right)$$

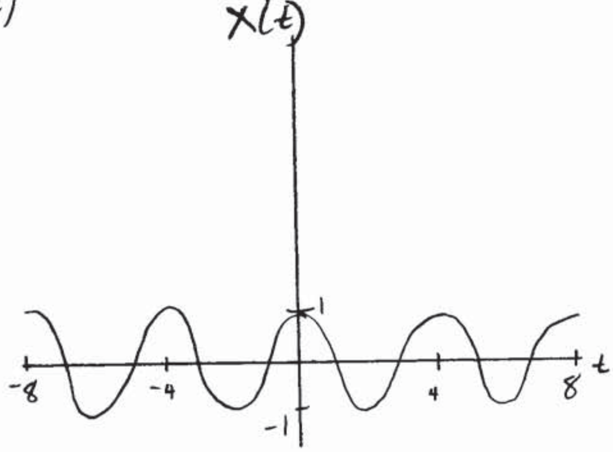
$T_0 = 8$

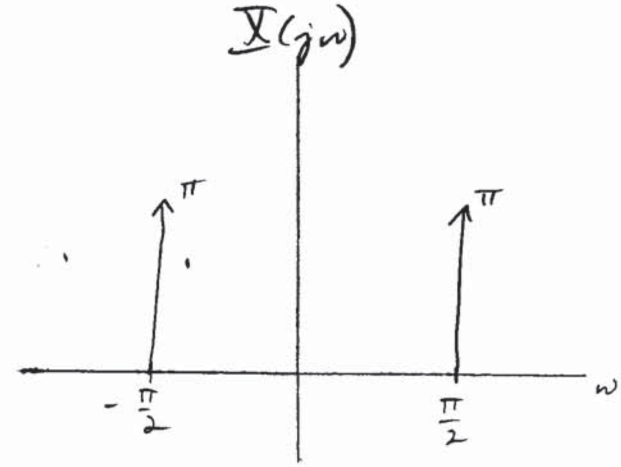
$$X[k] = \frac{4}{\pi} \, Area\left(X'\left(jk\frac{\pi}{4}\right)\right)$$

$$\frac{7\pi}{2} = k\frac{\pi}{4}$$

$$2 \cdot 7 = k = 14$$

3, i)

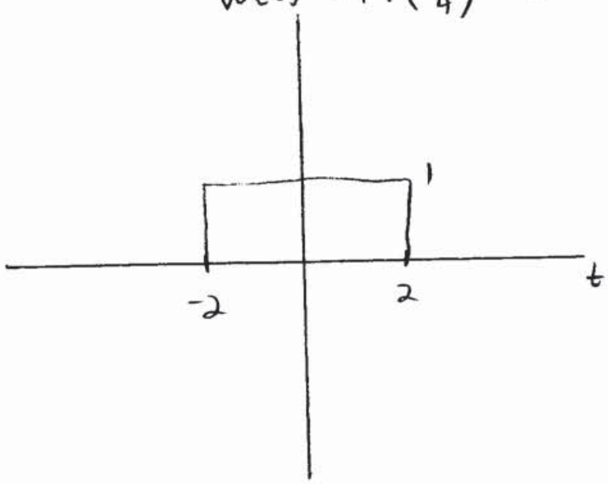


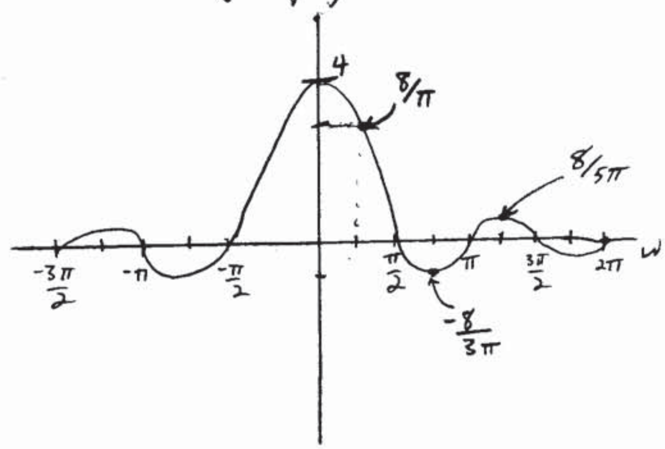


$$w(t) = \Pi\left(\frac{t}{4}\right) \longleftrightarrow \frac{2\sin(2w)}{w} = W(jw)$$

$2w = n\pi$

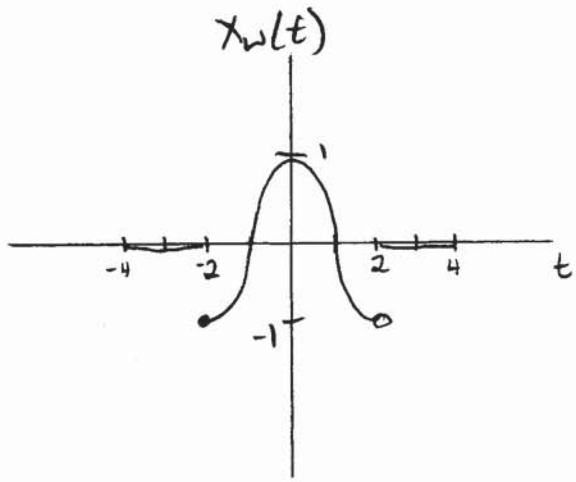
$w = \frac{n\pi}{2}$

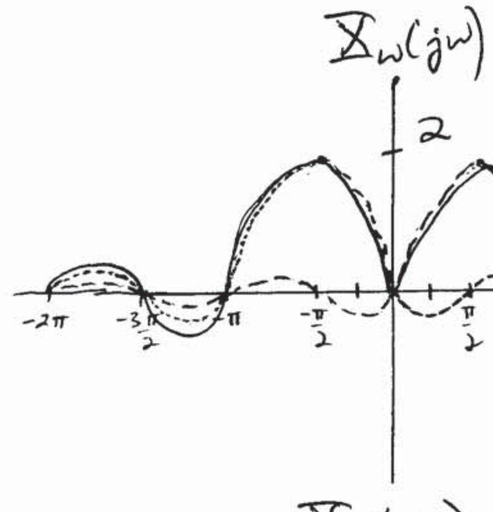




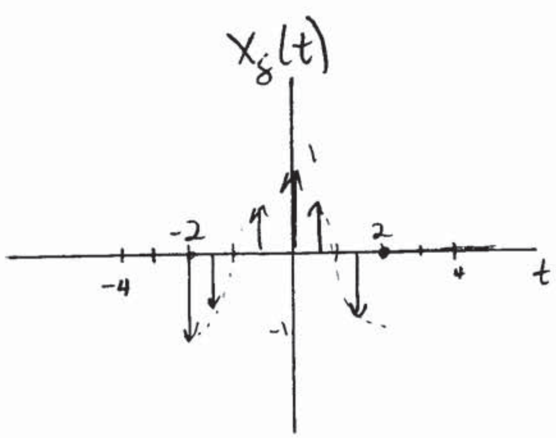
$\frac{4 \times \pi}{2\pi} = 2$

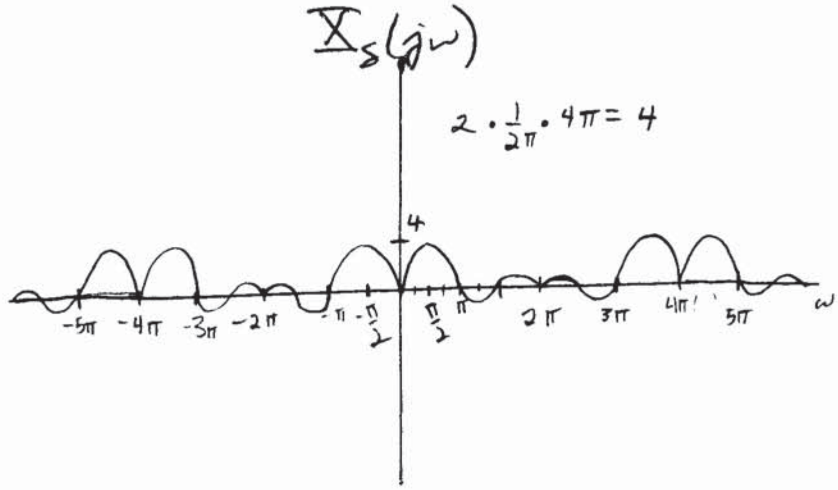
$\frac{48}{5\pi} = \frac{8}{\pi} + \frac{8}{5\pi} = \frac{8}{\pi}\left(\frac{6}{5}\right)$





$2 \cdot \frac{1}{2\pi} \cdot 4\pi = 4$





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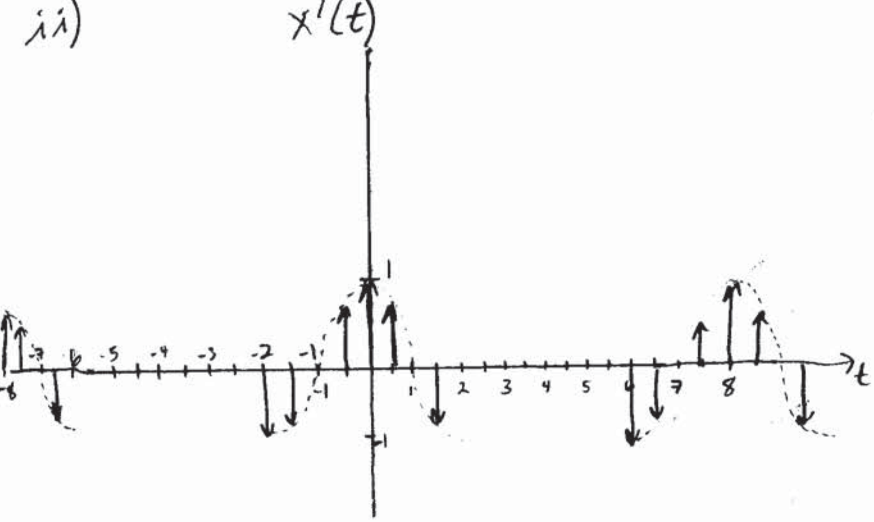
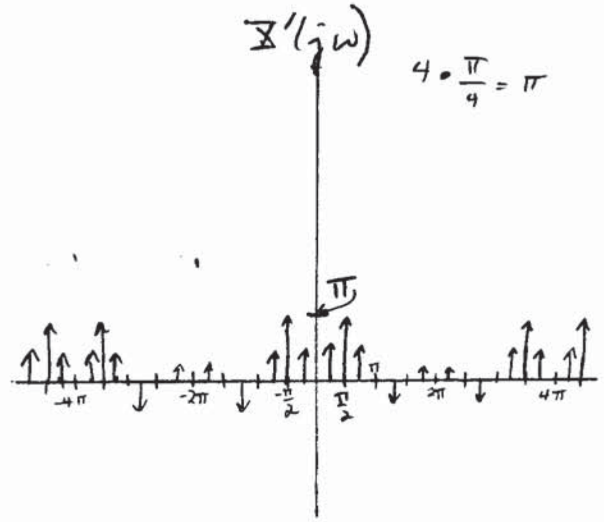
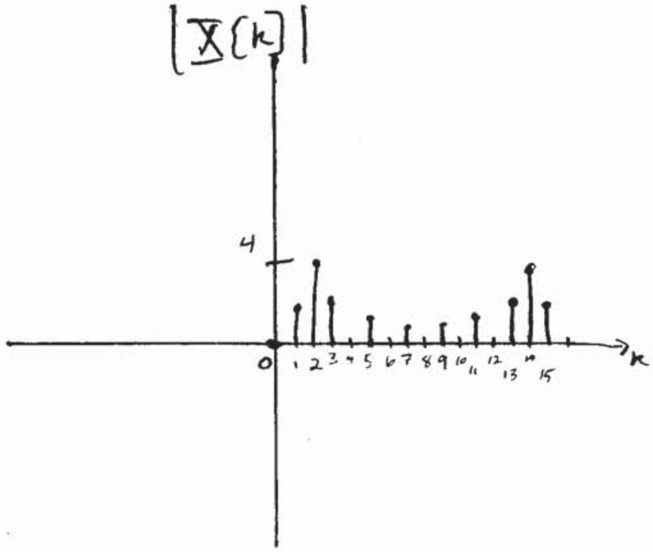
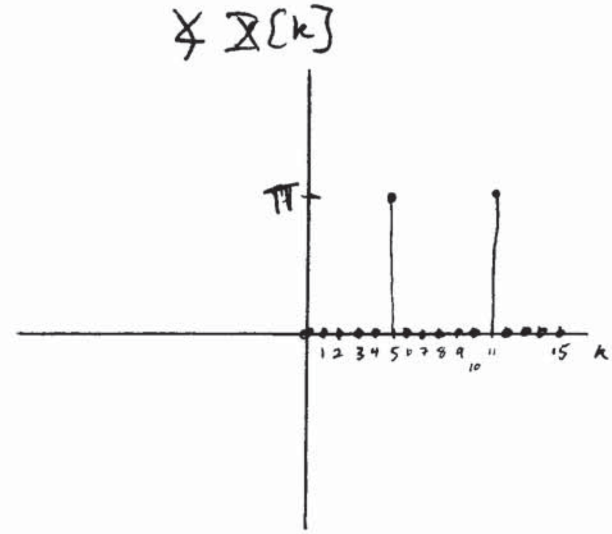




$4 \cdot \frac{\pi}{4} = \pi$





$$X[k] = \frac{T_0}{2\pi}\, Area\left(X'\left(jk\frac{2\pi}{T_0}\right)\right)$$

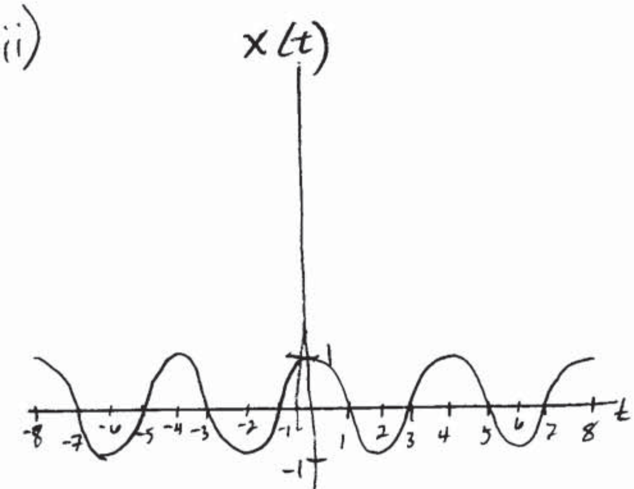
$T_0 = 8$

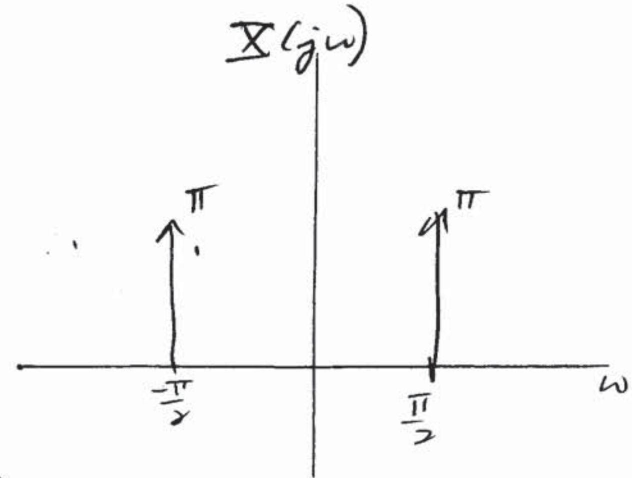
$$X[k] = \frac{4}{\pi}\, Area\left(X'\left(jk\frac{\pi}{4}\right)\right)$$

$$\frac{4}{\pi} \cdot \pi = 4$$

3

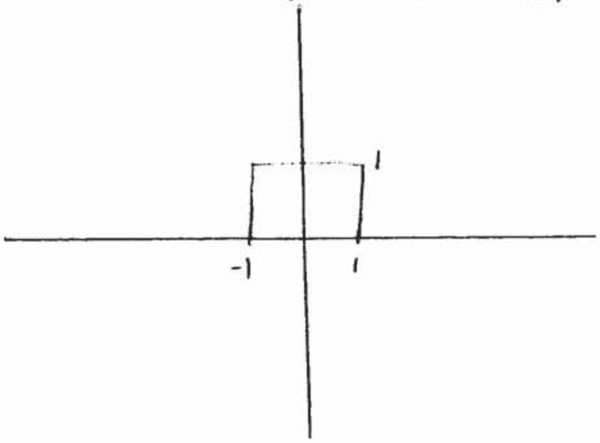
iii)

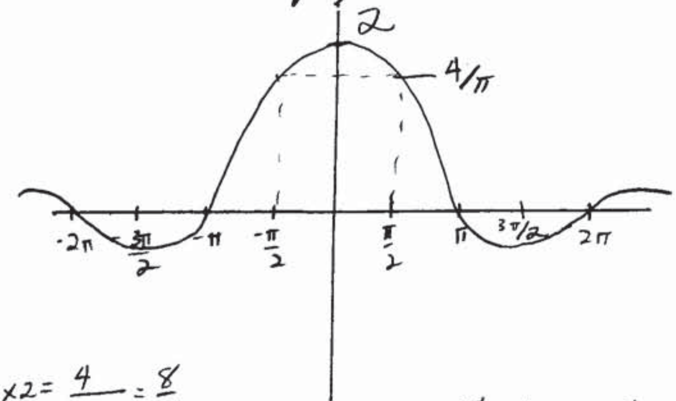




$W(t) = \Pi\left(\frac{4t}{T_0}\right) = \Pi\left(\frac{t}{2}\right) \rightarrow \frac{2\sin(\omega)}{\omega} = W(j\omega)$

$\omega = n\pi$

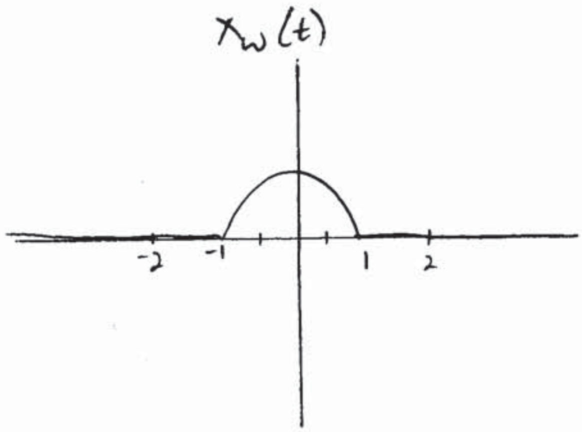


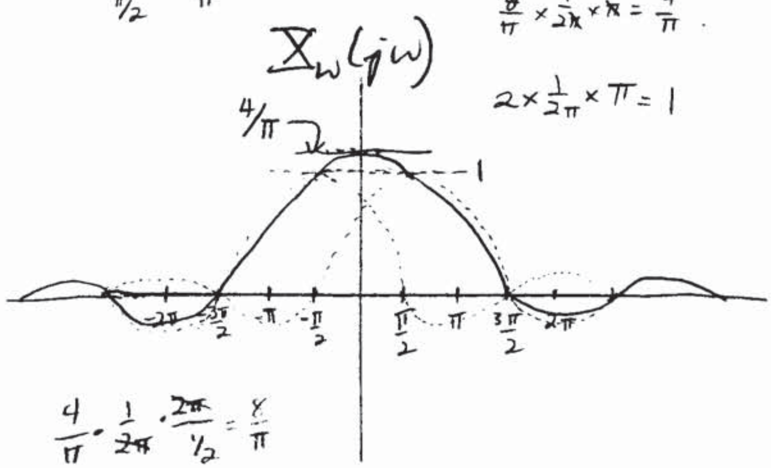


$\frac{2\sin\left(\frac{\pi}{2}\right)}{\pi/2} \times 2 = \frac{4}{\pi/2} = \frac{8}{\pi}$

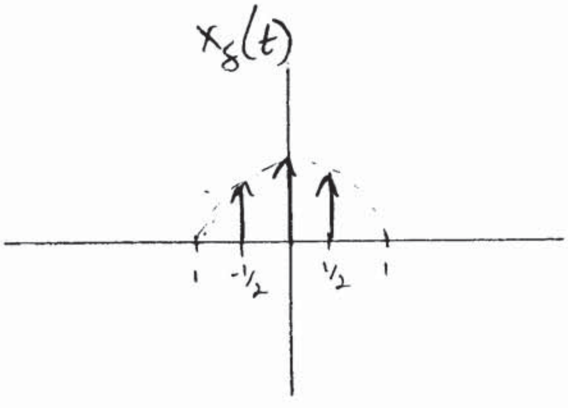
$\frac{8}{\pi} \times \frac{1}{2\pi} \times \pi = \frac{4}{\pi}$

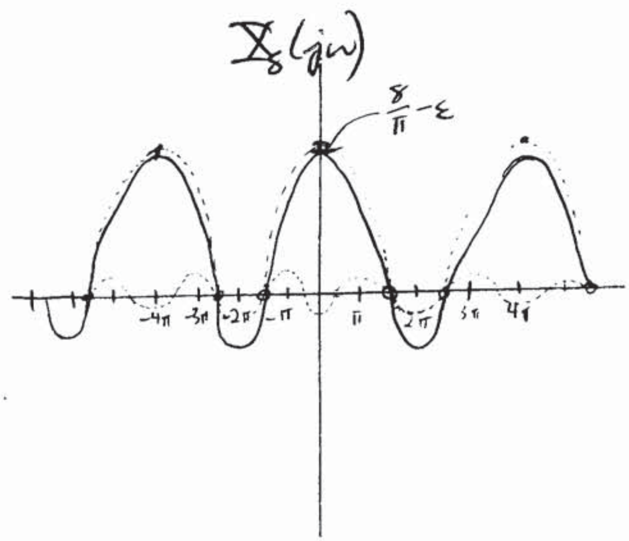
$2 \times \frac{1}{2\pi} \times \pi = 1$



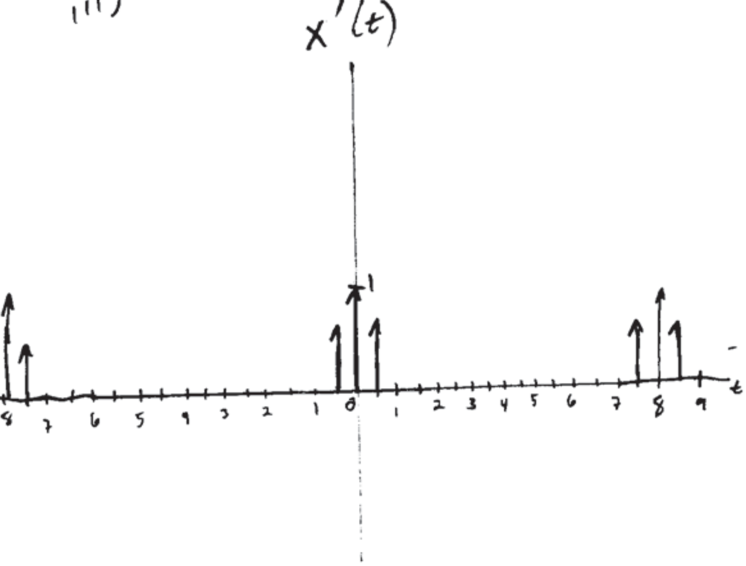


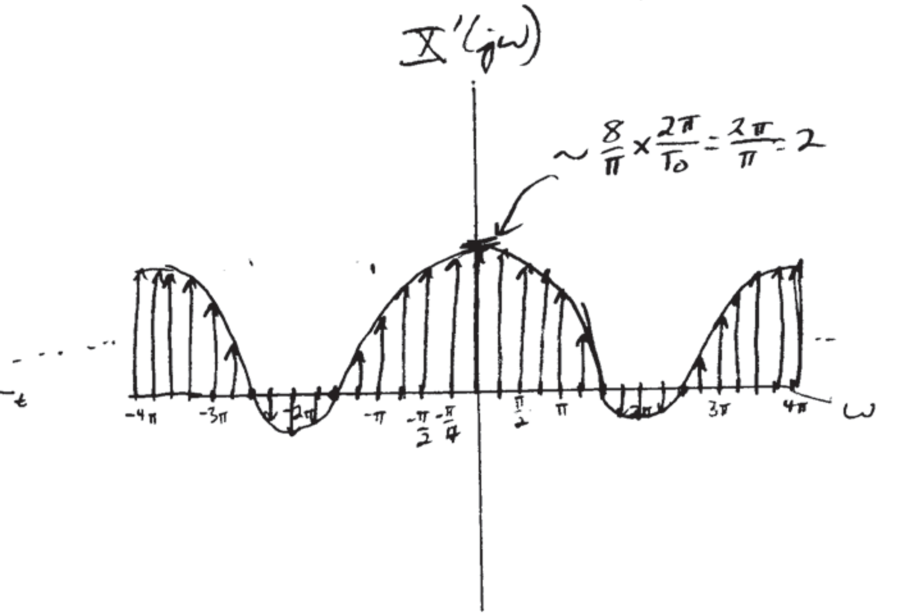
$\frac{4}{\pi} \cdot \frac{1}{\cancel{2\pi}} \cdot \frac{\cancel{2\pi}}{1/2} = \frac{8}{\pi}$

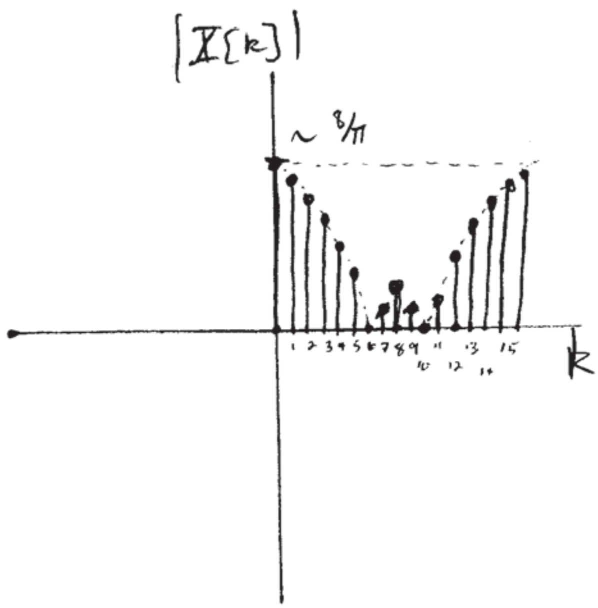


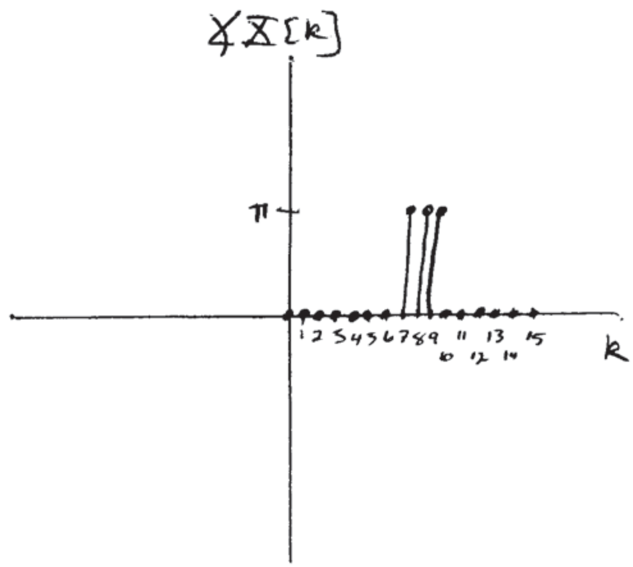


iii)







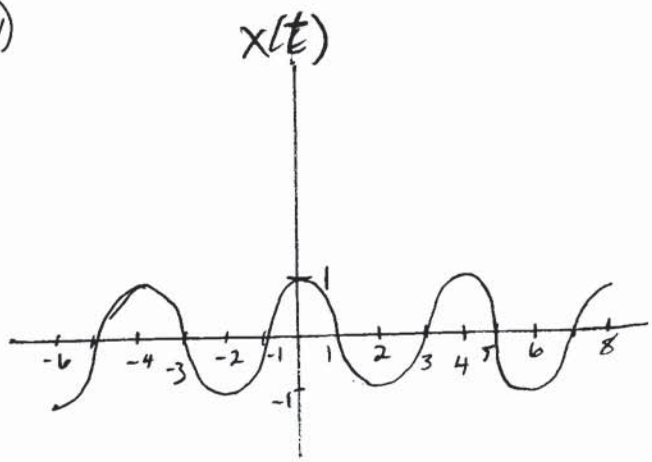


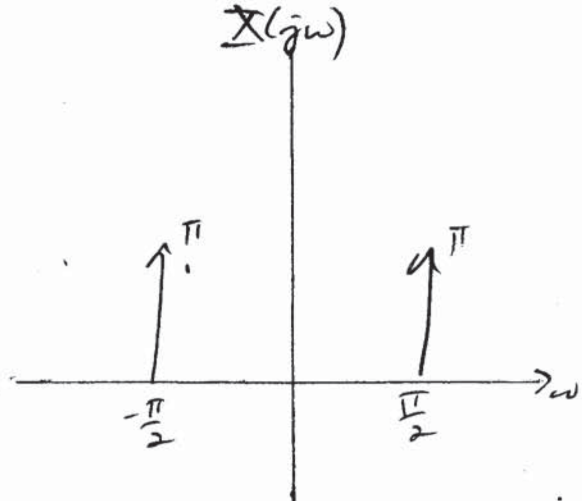
$$X[k] = \frac{T_0}{2\pi}\,\text{area}\left(X'\left(jk\frac{2\pi}{T_0}\right)\right)$$

$$= \frac{4}{\pi}\,\text{area}\left(X'\left(jk\frac{\pi}{4}\right)\right)$$

$$2 \cdot \frac{4}{\pi} = \frac{8}{\pi}$$

iv)

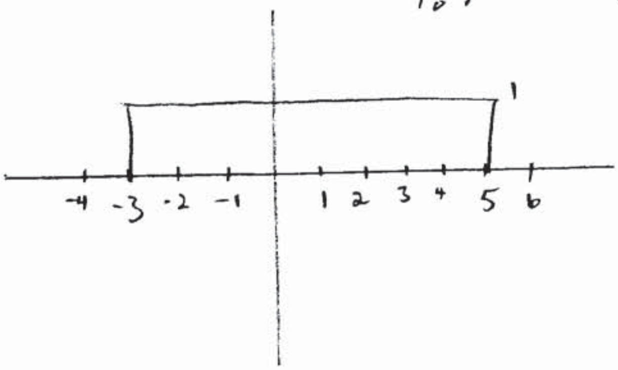


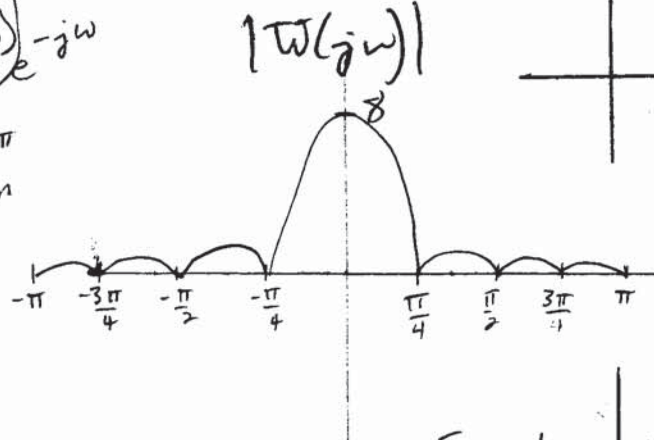


$w(t) = \Pi\left(\frac{t-1}{T_0}\right) \longleftrightarrow \left(\frac{2\sin(4w)}{w}\right)e^{-jw}$

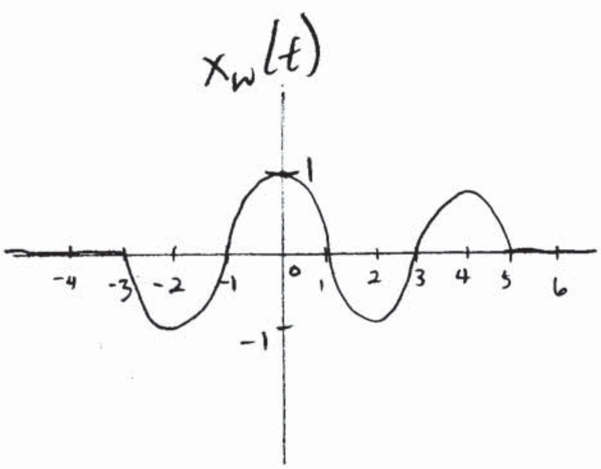
$4w = n\pi$

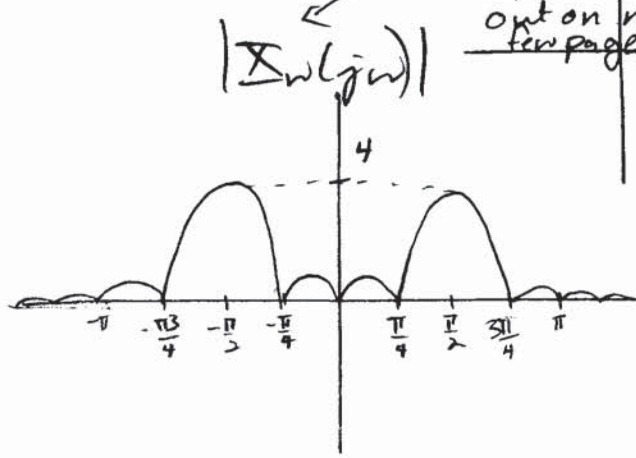
$w = \frac{\pi}{4}n$



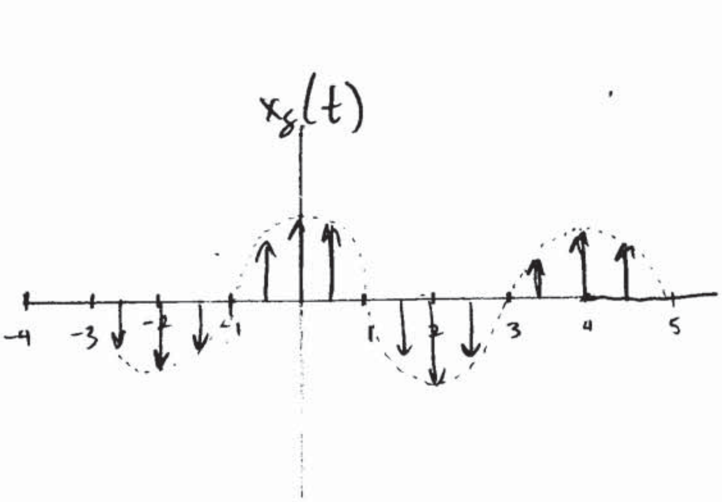


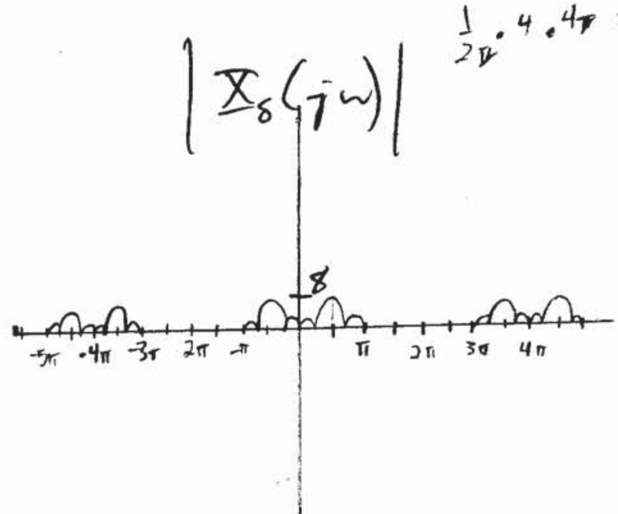
See phase worked out on next few pages





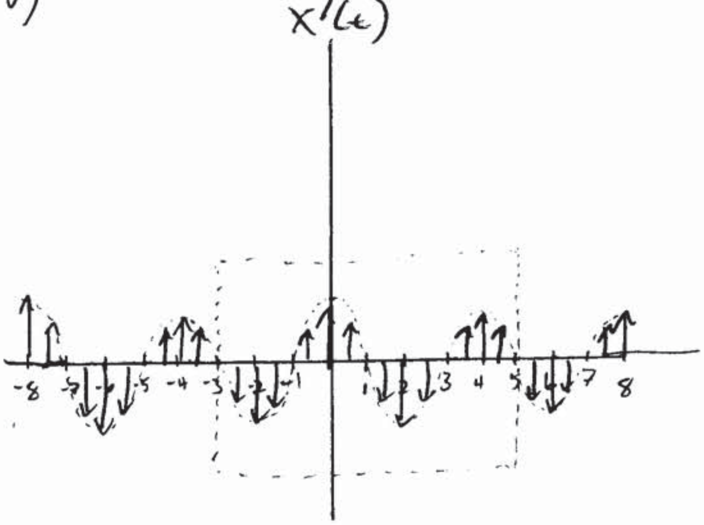
$\frac{1}{2\pi} \cdot 4 \cdot 4\pi = 8$

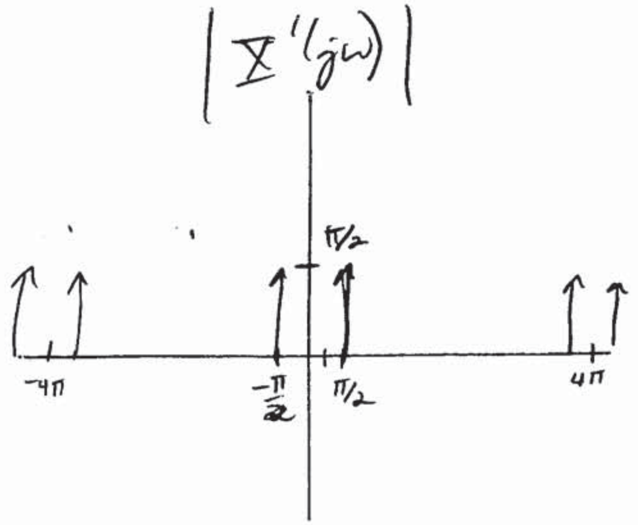


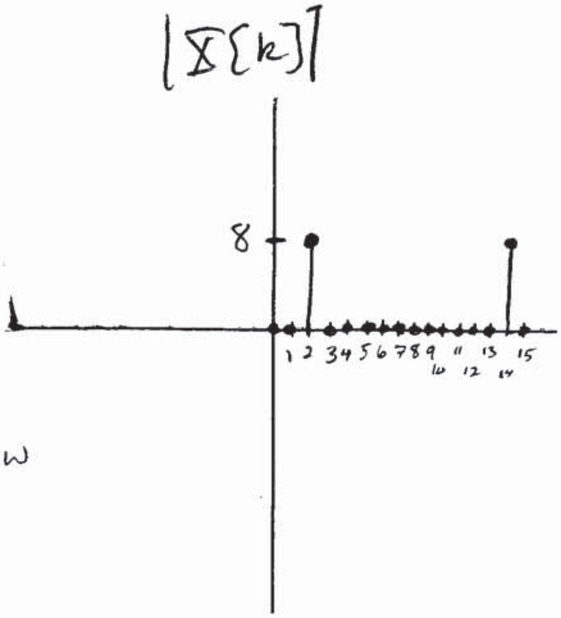


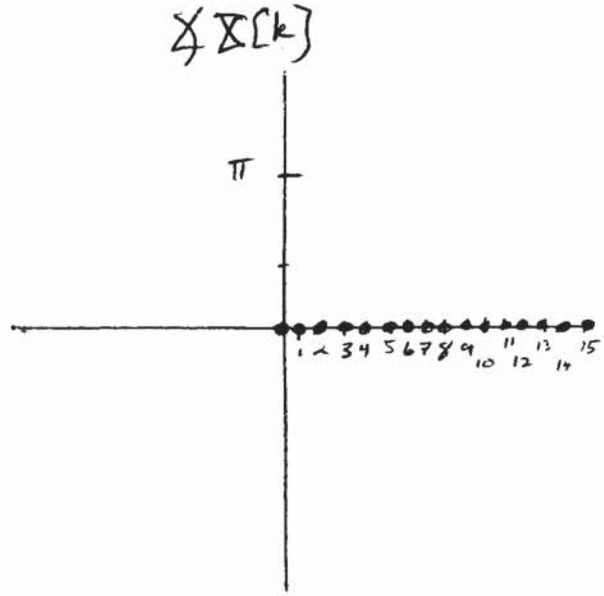
3

iv)









$\frac{\pi}{2} - \omega$

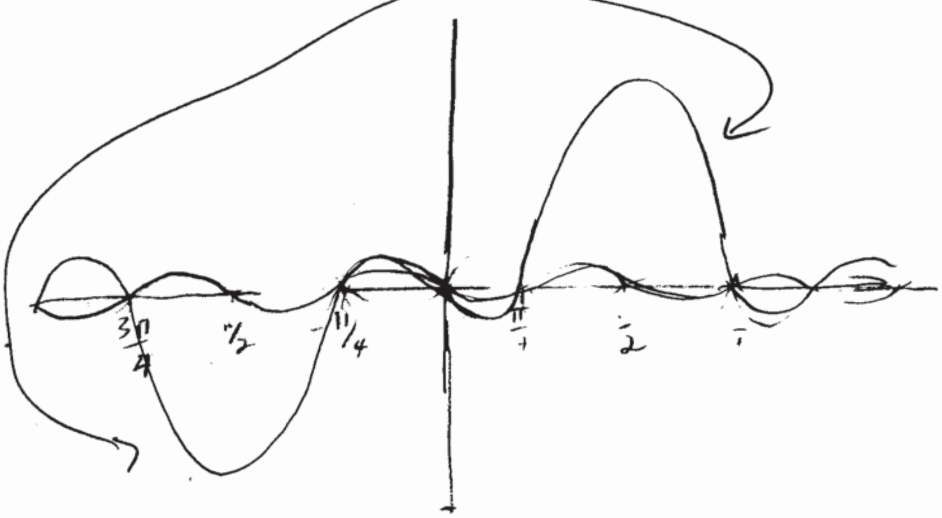
Magnitude for part iV)

$$X_w(jw) = \frac{1}{2}\left(\frac{2\sin\left(4\left(w-\frac{\pi}{2}\right)\right)}{\left(w-\frac{\pi}{2}\right)} e^{-j\left(w-\frac{\pi}{2}\right)} + \frac{2\sin\left(4\left(w+\frac{\pi}{2}\right)\right)}{\left(w+\frac{\pi}{2}\right)} e^{-j\left(w+\frac{\pi}{2}\right)}\right)$$

$$2X_w(jw) = e^{-jw}\left(\frac{2\sin\left(4\left(w-\frac{\pi}{2}\right)\right)}{\left(w-\frac{\pi}{2}\right)} \underbrace{e^{j\frac{\pi}{2}}}_{+j} + \frac{2\sin\left(4\left(w+\pi/2\right)\right)}{\left(w+\frac{\pi}{2}\right)} \underbrace{e^{-j\pi/2}}_{-j}\right)$$

$$2X_w(jw) = e^{-jw} j\left(\frac{2\sin\left(4\left(w-\frac{\pi}{2}\right)\right)}{w-\frac{\pi}{2}} - \frac{2\sin\left(4\left(w+\frac{\pi}{2}\right)\right)}{w+\frac{\pi}{2}}\right)$$

$$|X_w(jw)| = \frac{1}{2}\left|\underbrace{\frac{2\sin\left(4\left(w-\frac{\pi}{2}\right)\right)}{w-\pi/2}} - \underbrace{\frac{2\sin\left(4\left(w+\frac{\pi}{2}\right)\right)}{w+\pi/2}}\right|$$



$$e^{-jw} e^{j\frac{\pi}{2}} = e^{j\left(-w+\frac{\pi}{2}\right)}$$

Phase for part iv)

$$X_w(j\omega) = \frac{e^{j(\pi/2-\omega)}}{2}\left(\text{sinc}_1(\omega) - \text{sinc}_2(\omega)\right)$$

$$X_\delta(j\omega) = \frac{1}{2\pi} X_w(j\omega) * \frac{2\pi}{T_s} \sum_k \delta\left(\omega - \frac{2\pi k}{T_s}\right)$$

$$X_\delta(j\omega) = \sum_k 2 X_w(j\omega) * \delta\left(\omega - \frac{2\pi k}{T_s}\right)$$

$$= 2\sum_k X_w\left(j\left(\omega - \frac{2\pi k}{T_s}\right)\right) = 2\sum_k X_w\left(j(\omega - 4\pi k)\right)$$

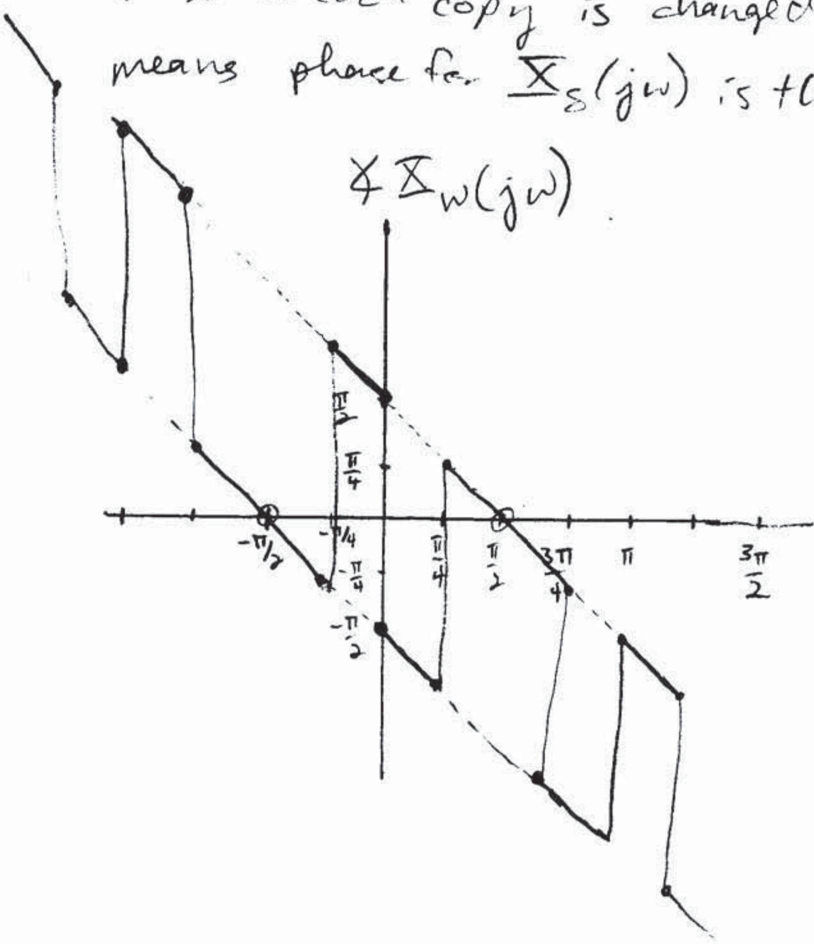
$$\cancel{\frac{1}{2}} e^{j\left(\frac{\pi}{2} - \omega\right)}\left(\frac{\text{sinc}_1(\omega) - \text{sinc}_2(\omega)}{2}\right)$$

$$+ e^{j\left(\frac{\pi}{2} - \omega + 4\pi\right)}\left(\frac{\text{sinc}_1(\omega - 4\pi) - \text{sinc}_2(\omega - 4\pi)}{2}\right)$$

$$+ \dots$$

Phase for each copy is changed by amount 4π $\rightarrow$ this means phase for $X_\delta(j\omega)$ is the same for $X_w(j\omega)$.

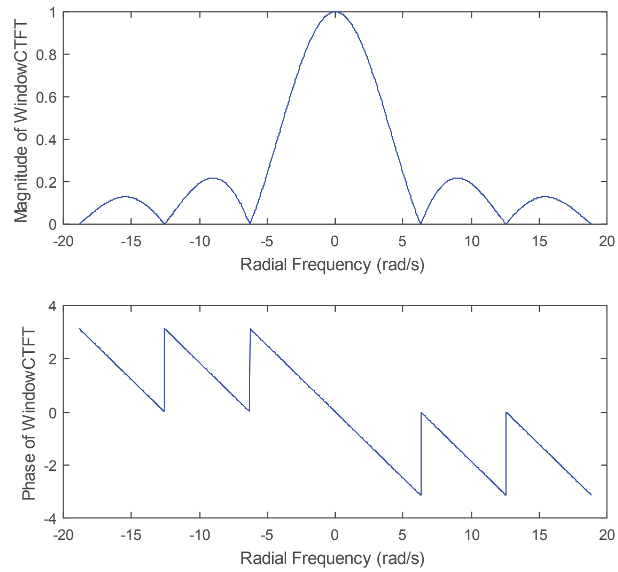
$\measuredangle X_w(j\omega)$



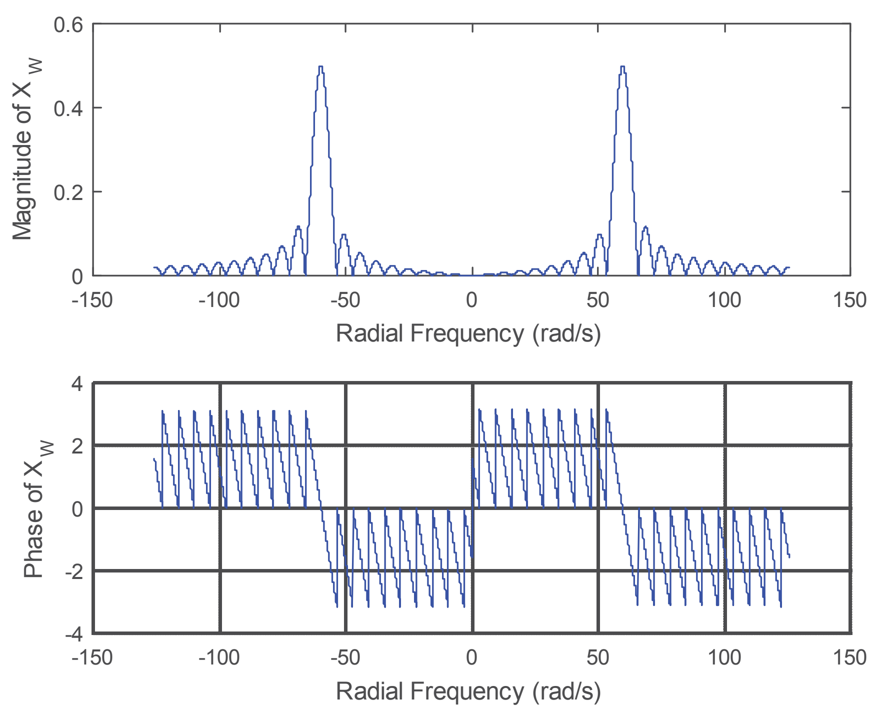
Problem 4

For this problem, we have a window between 0 seconds and 1 second. The window has a CTFT as shown below:

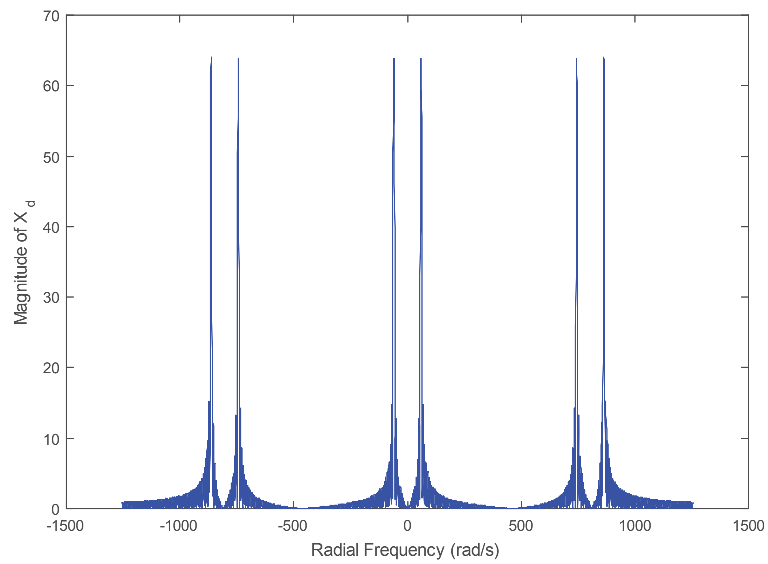
$$W(j\omega) = \frac{2\sin\left(\frac{\omega}{2}\right)}{\omega} e^{-\frac{j\omega}{2}}$$



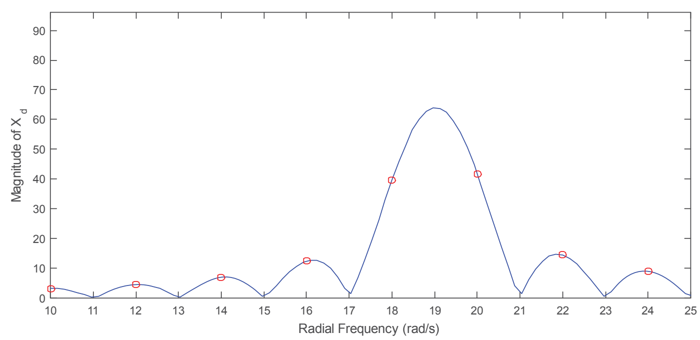
Note that the phase is linear, but with phase jumps each time $\frac{2\sin\left(\frac{\omega}{2}\right)}{\omega}$ changes sign. The signal has frequency $2\pi 9.5$ rad/s, so this window is shifted to the left and right by $2\pi 9.5$ rad/s and summed in complex value.

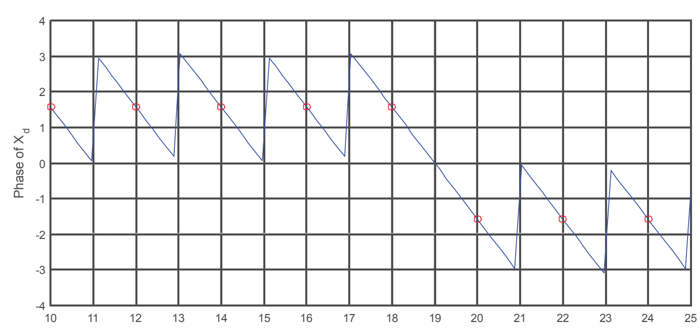


When $x_w(t)$ is sampled, we make the CTFT periodic:



These are pairs of sinc functions centered at $\frac{2\pi}{T_s} = 2\pi 128$ rad/s in ω. Next, we sample this periodic function in frequency, with a period of $\frac{2\pi}{T_0} = 2\pi$ rad/s.





<u>(Note that I have normalized the ω axis in the above plot by π to make the even multiples of π stand out from the odd multiples of π)</u>

Effect of window length: Since the sincs are centered on odd multiples of π, you can see that we are no longer sampling zeros of the sinc, but the center of each side-lobe. In general these values will be complex, so you can see that at each lobe the phase is non-zero. The residual energy around the actual frequency in the DFT will be as a result of the fact that there are a non-integer number of periods in the original signal that we measured.

Effect of window time shift: The complex value of the sampled CTFT comes from the fact that we had a time shift in the original window. *Note that if the window had been centered at t=0, the sampled CTFT would have been real-valued.*

Interpreting the real and imaginary part of the DFT: The phase of the DFT can be seen from taking the arctangent of imaginary part over real part:

$$\angle X[k] = \operatorname{atan}\left(\frac{Im\{X[k]\}}{Re\{X[k]\}}\right)$$

The imaginary part looks like a tangent function over k, so taking the arctangent will give a linear phase angle, which is consistent with our plot of phase above.