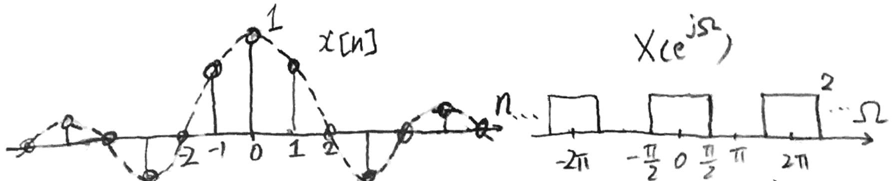
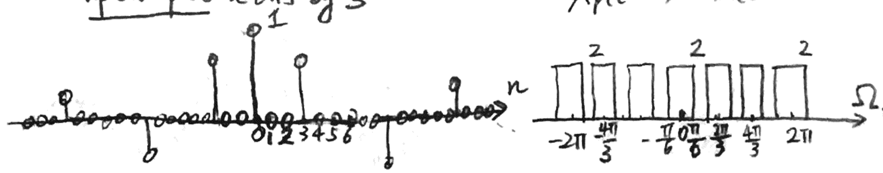
PS 9. Fall 2016 Ming Jin

Prob 1.

$x[n] = \dfrac{2\sin\frac{\pi n}{2}}{\pi n}, \longleftrightarrow X(e^{j\Omega}) = \begin{cases} 2 & |\omega| < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| < \pi \end{cases}$

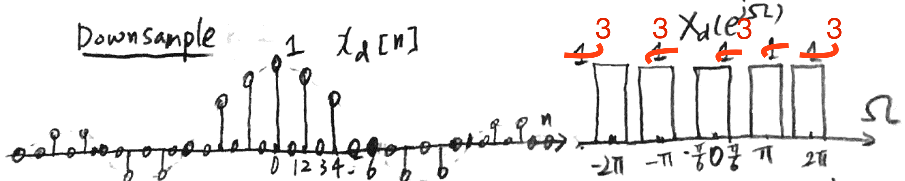


Upsample $x[n]$ by 3 $\quad X_p(e^{j\Omega}) = X(e^{j3\Omega})$

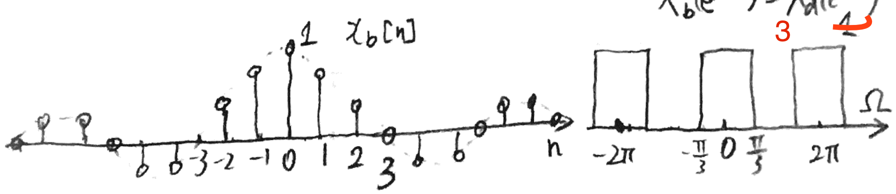


Interpolation $x_u[n] = x_p[n] * h[n]$ $\quad X_u(e^{j\Omega})$

Downsample $x_d[n]$ $\quad X_d(e^{j\Omega})$

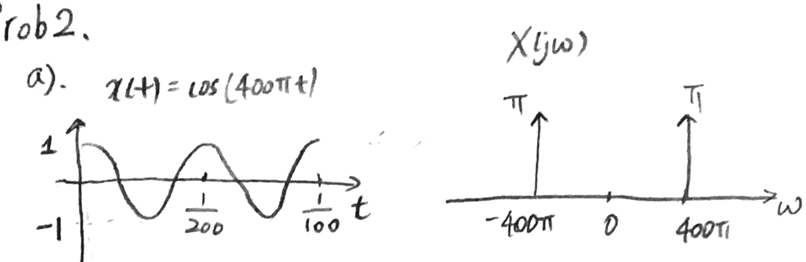


$X_b(e^{j2\Omega}) = X_d(e^{j\Omega})$

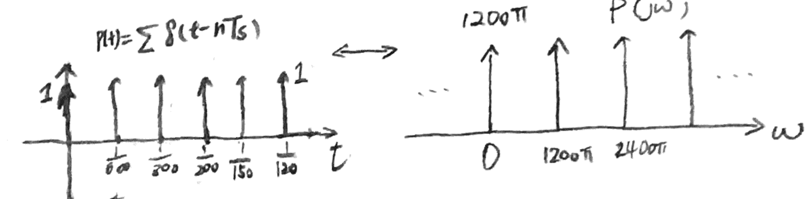


Prob 2.

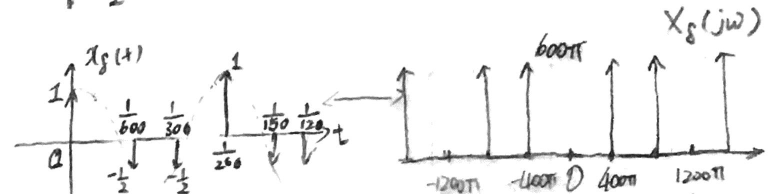
a). $x(t) = \cos(400\pi t)$ $\quad X(j\omega)$



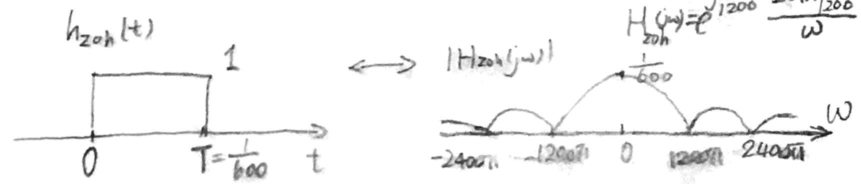
$p(t) = \sum \delta(t - nT_s) \longleftrightarrow P(j\omega)$



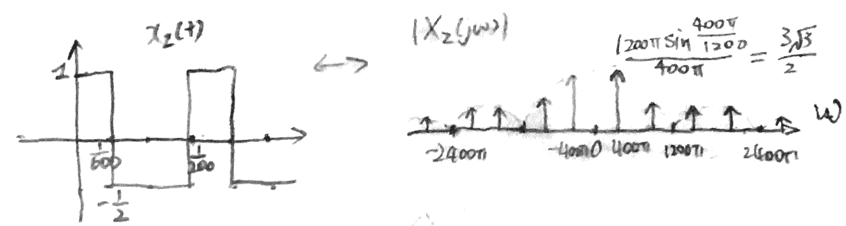
$x_\delta(t) \longleftrightarrow X_\delta(j\omega)$



$h_{zoh}(t) \longleftrightarrow |H_{zoh}(j\omega)|$ $\quad H_{zoh}(j\omega) = e^{-j\frac{\omega}{1200}} \dfrac{2\sin\frac{\omega}{1200}}{\omega}$



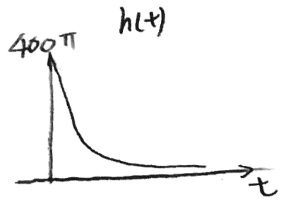
$x_z(t) \longleftrightarrow |X_z(j\omega)|$ $\quad \dfrac{1200\pi \sin\frac{400\pi}{1200}}{400\pi} = \dfrac{3\sqrt{3}}{2}$



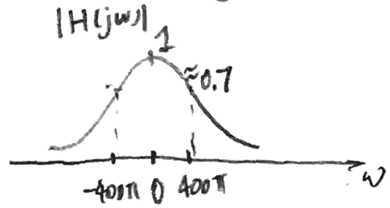
Prob2 continue

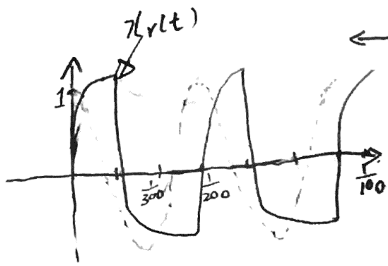
$h(t) = 400\pi e^{-400\pi t} u(t).$

$$H(j\omega) = \int h(t) e^{-j\omega t} dt = \int_0^\infty 400\pi e^{-400\pi t} e^{-j\omega t} dt = \frac{400\pi}{400\pi + j\omega}$$

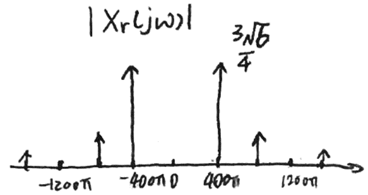


$\longleftrightarrow$





$\longleftrightarrow$



$X_r(j\omega) = X_z(j\omega) \cdot H(j\omega)$

b). Note that $x_r(t)$ is periodic w/ $\omega_0 = 400\pi$, so an impulse in $X_r(j\omega)$ at $\omega_k = 400\pi k$ w/ magnitude B_k corresponds to Fourier Series coefficient a_k w/ magnitude $\frac{B_k}{2\pi}$

$P_{fund} = |a_1|^2 + |a_{-1}|^2 = 2|a_1|^2 = 2 \cdot \left(\frac{B_1}{2\pi}\right)^2 = 2\left(\frac{3\sqrt{6}}{8\pi}\right)^2 \approx 0.1710$

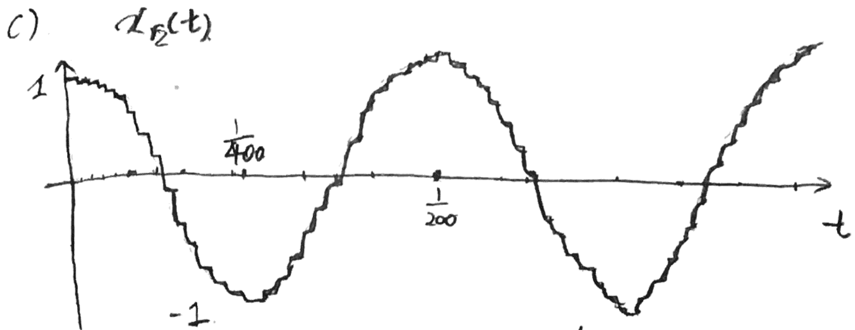
$|a_2| = \frac{1}{2\pi}|X_r(j800\pi)| = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{5}} \cdot \frac{1200\pi \sin\frac{800\pi}{1200}}{800\pi} \approx 0.0425$

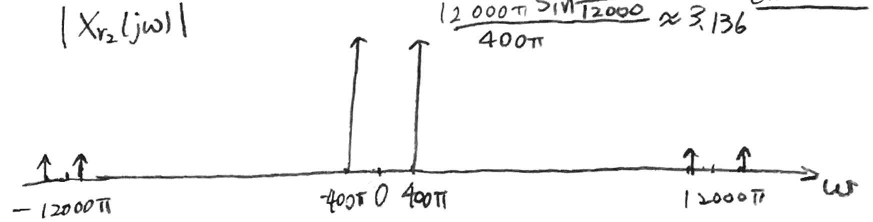
$|a_4| = \frac{1}{2\pi}|X_r(j1600\pi)| = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{17}} \cdot \frac{1200\pi \cdot \left|\sin\frac{1600\pi}{1200}\right|}{1600\pi} \approx 0.0251$

$P_{all} = \sum_k |a_k|^2 \approx 2|a_1|^2 + 2|a_2|^2 + 2|a_4|^2 \approx 0.1893$

Fraction of power not at ± 200Hz $= \frac{P_{all} - P_{fund}}{P_{all}} \approx 9.7\%$

c)





$\frac{12000\pi \sin\frac{400\pi}{12000}}{400\pi} \approx 3.136$ close to π

$P_{fund} = |a_1|^2 + |a_{-1}|^2 = 2|a_1|^2 = 2 \cdot \left(\frac{3.136}{2\pi}\right)^2 \approx 0.4982$

$|a_{29}| = \frac{1}{2\pi}|X_{r_2}(j400\pi \cdot 29)| = \frac{1}{2\pi} \frac{1}{\sqrt{842}} \frac{12000\pi \sin\frac{400\pi \cdot 29}{12000}}{400\pi \cdot 29} \approx 5.9 \times 10^{-4}$

$|a_{31}| = \frac{1}{2\pi}|X_{r_2}(j400\pi \cdot 31)| = \frac{1}{2\pi} \frac{1}{\sqrt{962}} \frac{12000\pi \left|\sin\frac{400\pi \cdot 31}{12000}\right|}{400\pi \cdot 31} \approx 5.19 \times 10^{-4}$

$P_{all} = \sum_k |a_k|^2 \approx 2|a_1|^2 + 2|a_{29}|^2 + 2|a_{31}|^2 \approx 0.4982$

Fraction of power not at ± 200Hz ≈ 0

It is a high-fidelity reconstruction. The fraction of power not at ± 200 Hz is about ~~255dB~~.

$\approx \frac{2 \times (5\times10^{-4})^2}{.5} = \frac{5\times10^{-7}}{5\times10^{-1}}$

$= 10^{-6} \Rightarrow 10\log_{10} 10^{-6}$

$= -60$dB

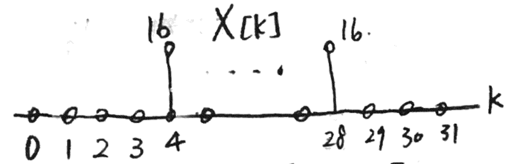
Prob. 3.

a) $x(t) = \cos(2\pi \cdot 1000 t)$

$x[n] = x(nT) = \cos(2\pi \cdot 1000 \cdot n \frac{1}{8000}) = \cos(\frac{\pi}{4} n) = \frac{1}{2} e^{j\frac{\pi}{4}n} + \frac{1}{2} e^{-j\frac{\pi}{4}n}$

By DFT, $x[n] = \frac{1}{32} \sum_{k=0}^{31} X[k] e^{j\frac{\pi}{16}nk}$

By inspection, we should have $X[k] = \begin{cases} 16, & k = 4, 28 \\ 0, & \text{o.w.} \end{cases}$

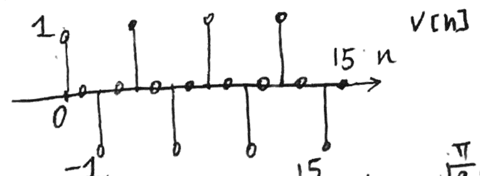


Spacing of samples: $\frac{2\pi}{NT} = \frac{2\pi}{32 \cdot \frac{1}{8000}} = 500\pi$ rad/sec

$k = 3$ corresponds to $3 \cdot 500\pi = 1500\pi$ rad/sec

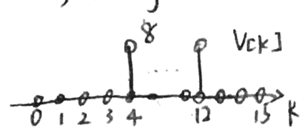
b). $v[n]$ is equivalently given by sampling $x(t)$ by 4KHz.

$v[n] = x(2nT) = x[2n] = \cos(\frac{\pi}{2} n) = \frac{1}{2} e^{j\frac{\pi}{2}n} + \frac{1}{2} e^{-j\frac{\pi}{2}n}$



By DFT, $v[n] = \frac{1}{16} \sum_{k=0}^{15} V[k] e^{j\frac{\pi}{8}nk}$, therefore.

$V[k] = \begin{cases} 8, & k = 4, 12 \\ 0, & \text{o.w.} \end{cases}$



Spacing: $\frac{2\pi}{NT_2} = \frac{2\pi}{16 \cdot \frac{1}{4000}} = 500\pi$ rad/sec

The spacings are the same since our window length ($\frac{16}{4000}$ sec) does not change!

c). $Y[k] = \sum_{n=0}^{63} y[n] e^{-j\frac{2\pi}{64}kn}$

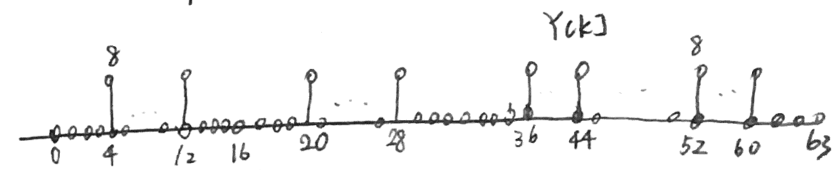
$= \sum_{n=0}^{63} v[\frac{n}{4}] e^{-j\frac{\pi}{32}kn}$, $m = \frac{n}{4}$

n multiples of 4

$= \sum_{m=0}^{15} v[m] e^{-j\frac{\pi}{8}km}$

$= V[k]$, $k = 0, 1, \cdots, 15$ and

$Y[k] = Y[k+16]$



The spacing is $\frac{2\pi}{N_3 T_3} = \frac{2\pi}{64 \cdot \frac{1}{16000}} = 500\pi$ rad/sec

(d). Since $z[n] = x(nT_3) = \cos(2\pi \cdot 1000 \cdot n \frac{1}{16000}) = \frac{1}{2} e^{j\frac{\pi}{8}n} + \frac{1}{2} e^{-j\frac{\pi}{8}n}$

By DFT, $z[n] = \frac{1}{64} \sum_{k=0}^{63} Z[k] e^{j\frac{\pi}{32}nk}$, therefore,

$Z[k] = \begin{cases} 32, & k = 4, 60 \\ 0, & \text{o.w.} \end{cases}$

We should have $H[k]$ to extract only $k = 4, 60$

prob 3 continue.

One example is $H[k] = \begin{cases} 4, & 0 \le k \le 8,\ 55 \le k \le 63 \\ 0 & \text{o.w} \end{cases}$

Using the Inverse DFT:

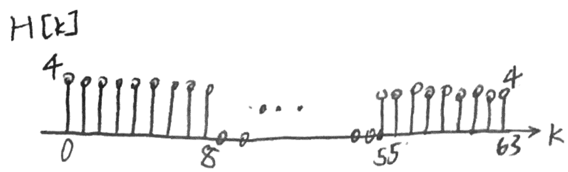
$$h[n] = \frac{1}{64} \sum_{k=0}^{63} H[k] e^{j2\pi \frac{kn}{64}}$$

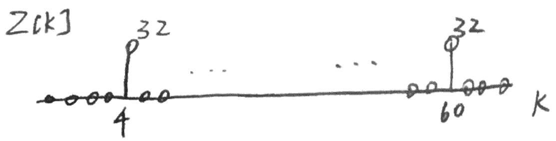
$$= \frac{1}{64}\left[\sum_{k=0}^{8} 4 \cdot e^{j2\pi\frac{kn}{64}} + \sum_{k=55}^{63} 4 \cdot e^{j2\pi\frac{kn}{64}}\right]$$

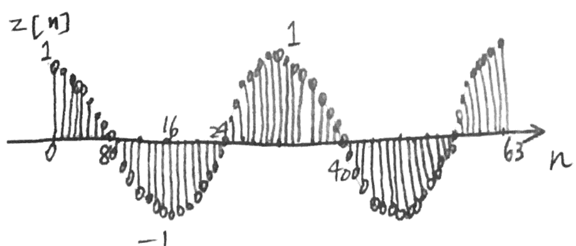
$$= \frac{1}{16} \sum_{k=-8}^{8} e^{j2\pi\frac{kn}{64}} \quad (H[k] = H[k+64] = H[k-64])$$

$$= \frac{1}{16} e^{-j\frac{\pi n}{4}} \frac{1-e^{j\frac{\pi n}{2}}}{1-e^{j\frac{2\pi n}{64}}}$$

$$= \frac{1}{16} e^{-j\frac{\pi n}{64}} \frac{\sin\frac{\pi n}{4}}{\sin\frac{\pi n}{64}}$$







The spacing of samples is $\frac{1}{16000}$ sec since we sample @ 16kHz

Prob4.

i. For $x(t)=te^{-2t}u(t)$,

$X(s)=\int_{-\infty}^{\infty} te^{-2t}u(t)e^{-st}dt = \int_0^\infty te^{-(2+s)t}dt$ finite only $\sigma>-2$

(integration by parts).

$= -\frac{1}{2+s}\int_0^\infty t\,de^{-(2+s)t} = -\frac{1}{2+s}\left[te^{-(2+s)t}\Big|_0^\infty - \int_0^\infty e^{-(2+s)t}dt\right]$

$= -\frac{1}{2+s}\left[\frac{1}{2+s}e^{-(2+s)t}\Big|_0^\infty\right] = \frac{1}{(2+s)^2}$

finite only if $\sigma>-2$

ROC: $s>-2$, poles $s=-2$

for $h(t)=e^{+2t}u(t)$,

$H(s)=\int_{-\infty}^{\infty} e^{+2t}u(t)e^{-st}dt = \int_0^\infty e^{(2-s)t}dt$

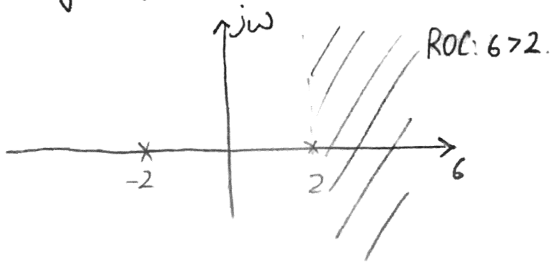
$= \frac{1}{2-s}e^{(2-s)t}\Big|_0^\infty = \frac{1}{s-2}$

finite only when $\sigma>2$

ROC: $s>2$, poles $s=2$

Since $y(t)=x(t)*h(t)$, $Y(s)=X(s)H(s)=\frac{1}{(2+s)^2(s-2)}$

Therefore the poles are at $s=-2,+2$, and no zeros, the region of convergence (ROC) is $\sigma>2$



ii. $x(t)=e^{-6t}u(t)$, $X(s)=\frac{1}{s+6}$, ROC: $\sigma>-6$

from the result in i

$h(t)=\sin(4\pi t)$,

$H(s)=\int_0^\infty \sin(4\pi t)e^{-st}dt = \int_0^\infty \frac{1}{2j}(e^{j4\pi t}-e^{-j4\pi t})e^{-st}dt$

$=\frac{1}{2j}\left[\int_0^\infty e^{(-s+j4\pi)t}dt - \int_0^\infty e^{(-s-j4\pi)t}dt\right]$

converge for $Re\{s\}>0$

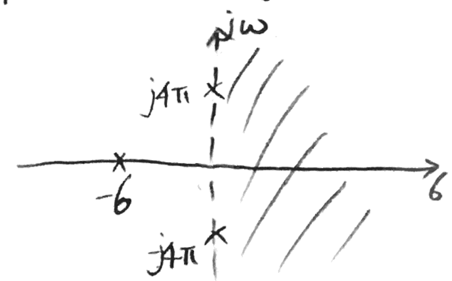
$=\frac{1}{2j}\left[\frac{1}{-s+j4\pi}e^{(-s+j4\pi)t}\Big|_0^\infty + \frac{1}{s+j4\pi}e^{-(s+j4\pi)t}\Big|_0^\infty\right]$

$=\frac{1}{2j}\left[\frac{1}{s-j4\pi}-\frac{1}{s+j4\pi}\right]=\frac{4\pi}{s^2+16\pi^2}$

ROC: $\sigma>0$, poles at $s=\pm j4\pi$.

$Y(s)=X(s)H(s)=\frac{1}{s+6}\,\frac{4\pi}{s^2+16\pi^2}$

poles at $s=-6,\pm j4\pi$, ROC: $\sigma>0$



prob 5.

In general, since $y(t) = h(t) * x(t)$, let $H(s)$, $Y(s)$, $X(s)$ denote the Laplace Transform of $h(t)$, $y(t)$, $x(t)$, then $Y(s) = H(s)X(s)$.

i) $Y(s) = \underbrace{\frac{1}{s+1}}_{X(s)} \underbrace{\frac{1}{s}}_{H(s)} = \frac{1}{s(s+1)} \Leftrightarrow y(t) = (1-e^{-t})u(t)$

ii) $X(s) = \frac{1}{s+2}$, since $h(t) = e^{-3} e^{-3(t-1)} u(t-1) = e^{-3} h'(t-1)$

where $h'(t) = e^{-3t} u(t)$. $H'(s) = \frac{1}{s+3}$.

By the Laplace property, $h'(t-1) \Leftrightarrow e^{-s} H'(s) = \frac{e^{-s}}{s+3}$

Note: this is valid since $h'(t-1) = 0$ for $0 \le t < 1$

Therefore, $H(s) = e^{-3}(e^{-s} H'(s)) = \frac{e^{-3-s}}{s+3}$

$Y(s) = X(s)H(s) = \frac{e^{-3-s}}{(s+2)(s+3)}$

Let $Y'(s) = Y(s-2) = \frac{e^{-3-s+2}}{s(s+1)} \Rightarrow y'(t) = e^{-1}(1-e^{-t+1})u(t-1)$

(using results from (i))

$\Rightarrow Y'(s+2) = Y(s)$

$\Rightarrow y(t) = e^{-2t} y'(t) = e^{-1-2t}(1-e^{-t+1})u(t-1)$

iii) $X(s) = \frac{1}{s}$, $H(s) = \frac{s}{s^2+(8\pi)^2}$

$Y(s) = X(s)H(s) = \frac{1}{8\pi} \frac{8\pi}{s^2+(8\pi)^2}$

Therefore, $y(t) = \frac{1}{8\pi} \sin 8\pi t \, u(t)$

Note for iv):
$\frac{s}{s+2} e^{-\frac{1}{2}s} = e^{-\frac{1}{2}s} - \frac{2}{s+2} e^{-\frac{1}{2}s}$
$\Leftrightarrow \delta(t-\tfrac{1}{2}) - 2e^{-2(t-\frac{1}{2})} u(t-\tfrac{1}{2})$

iv) $Y(s) = \underbrace{\left(\frac{1}{s+2} + \frac{1}{s^2}\right)}_{X(s)} \underbrace{\left(e^{-s} + s e^{-\frac{1}{2}s}\right)}_{H(s)}$

$= \frac{1}{s+2} e^{-s} + \frac{1}{s^2} e^{-s} + \frac{s}{s+2} e^{-\frac{1}{2}s} + \frac{1}{s} e^{-\frac{1}{2}s}$

$\Rightarrow y(t) = e^{-2(t-1)} + (t-1)u(t-1) + \delta(t-\tfrac{1}{2}) - 2e^{-2(t-\frac{1}{2})} u(t-\tfrac{1}{2}) + u(t-\tfrac{1}{2})$

v). $Y(s) = \underbrace{\frac{s}{s^2+(2\pi)^2}}_{X(s)} \underbrace{\frac{1}{s^2}}_{H(s)} = \frac{1}{s^2+(2\pi)^2} \frac{1}{s} = \frac{1}{2\pi} \frac{2\pi}{s^2+(2\pi)^2} \frac{1}{s}$

Let $Y'(s) = \frac{2\pi}{s^2+(2\pi)^2}$, $y'(t) = \sin 2\pi t$

Since $Y(s) = \frac{1}{2\pi} Y'(s) \frac{1}{s}$, using the Laplace transform prope.

$y(t) = \frac{1}{2\pi} \int_0^t y'(\tau)\, d\tau = \frac{1}{2\pi} \int_0^t \sin(2\pi\tau)\, d\tau$

$= -\frac{1}{(2\pi)^2} \cos 2\pi\tau \Big|_0^t = \frac{1}{(2\pi)^2}(1-\cos 2\pi t)\, u(t)$