EE120 Fall 2016

PS10 Solutions

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1 $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = x$

$\frac{dy}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - y(0^-) = sY(s) - 1$

$\frac{d^2y}{dt^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0^-) - \dot{y}(0^-) = s^2Y(s) - s - 2$

$s^2Y(s) - s - 2 + 2sY(s) - 2 - 3Y(s) = X(s)$

$X(s) = \mathcal{L}\{e^{-t}u(t)\} = \frac{1}{s+1} \quad \sigma > -1$

$Y(s)[s^2 + 2s - 3] = \frac{1}{s+1} + s + 4 = \frac{1 + s^2 + 5s + 4}{s+1}$

$Y(s) = \frac{s^2+5s+5}{(s+1)(s^2+2s-3)} = \frac{s^2+5s+5}{(s+1)(s-1)(s+3)}$

$Y(s) = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+3}$

$s^2+5s+5 = A(s-1)(s+3) + B(s+1)(s+3) + C(s+1)(s-1)$

$A(-2)(2) = 1 - 5 + 5 \rightarrow \boxed{A = -\frac{1}{4}}$

$B(2)(4) = 1 + 10 \rightarrow \boxed{B = \frac{11}{8}}$

$C(-2)(-4) = 9 - 15 \rightarrow \boxed{C = \frac{-1}{8}}$

$Y(s) = -\frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{11}{8}\left(\frac{1}{s-1}\right) - \frac{1}{8}\left(\frac{1}{s+3}\right)$

$\therefore \boxed{y(t) = u(t)\left[-\frac{1}{4}e^{-t} + \frac{11}{8}e^{t} - \frac{1}{8}e^{-3t}\right]}$

Note: the ZIR for $Y(s)$ is:

$\boxed{Y_{ZIR}(s) = \frac{s+4}{(s+3)(s-1)} = \frac{s^2+5s+4}{(s+3)(s-1)(s+1)}}$

The ZSR for $Y(s)$:

$\boxed{Y_{ZSR}(s) = \frac{1}{(s+3)(s-1)(s+1)}}$

$\therefore Y(s) = Y_{ZIR}(s) + Y_{ZSR}(s)$

$Y(s) = \frac{s^2+5s+4+1}{(s+3)(s-1)(s+1)}$

2 a)

$$\frac{1}{3}(4M+m)L(s^2\Theta(s)-\dot{\theta}(0^-)-s\theta(0^-)) = (m+M)g\,\Theta(s)-F(s)$$

$$F(s)-\frac{1}{3}(4M+m)L(s\theta(0^-)+\dot{\theta}(0^-)) = \Theta(s)\left(g(m+M)-\frac{1}{3}(4M+m)Ls^2\right)$$

$$\Theta(s)=\frac{F(s)-\frac{1}{3}(4M+m)L(s\theta(0^-)+\dot{\theta}(0^-))}{g(m+M)-\frac{1}{3}(4M+m)Ls^2}$$

$$\boxed{\Theta_{ZIR}(s)=\frac{-\frac{1}{3}(4M+m)L(s\theta(0^-)+\dot{\theta}(0^-))}{g(m+M)-\frac{1}{3}(4M+m)Ls^2}}$$

$$\boxed{\Theta_{ZSR}(s)=\frac{F(s)}{g(m+M)-\frac{1}{3}(4M+m)Ls^2}}$$

b) if we let $f(t)=\alpha L\theta(t)$,

$$F(s)=\alpha L\Theta(s)$$

$$\frac{1}{3}(4M+m)L(\Theta(s)s^2-\dot{\theta}(0^-)-s\theta(0^-)) = (m+M)g\,\Theta(s)-\alpha L\Theta(s)$$

$$=\Big[\underbrace{(m+M)g-\alpha L}_{\text{Let this be } A}\Big]\Theta(s)$$

Let $\frac{1}{3}(4M+m)L=B$

$$Bs^2\Theta(s)-A\Theta(s)=B\dot{\theta}(0^-)+Bs\theta(0^-)$$

$$\Theta(s)(Bs^2-A)=B\dot{\theta}(0^-)+Bs\theta(0^-)$$

$$\Theta(s)=\frac{B\dot{\theta}(0^-)+Bs\theta(0^-)}{Bs^2-A}=\frac{\dot{\theta}(0^-)+s\theta(0^-)}{s^2-\frac{A}{B}}$$

Where are the poles of $\Theta(s)$?

s_p @ $s_p^2=\frac{A}{B}$ $\quad s_p=\sqrt{\frac{A}{B}}(\pm 1)$

If $\frac{A}{B}>0$, s_p ~~is~~ are real, one is in RHP
$\therefore$ unstable

If $\frac{A}{B}<0$, s_p are complex conjugate pairs ~~and~~ s_p are on Imaginary axis.
$\therefore$ unstable

$\boxed{\text{No value of } \alpha \text{ will lead to stable system.}}$

2b) Check:

$$\frac{A}{B} = \frac{(m+M)g - \alpha L}{\frac{1}{3}(4M+m)L}$$

If $\frac{A}{B} > 0$, $(m+M)g - \alpha L > 0$

$$\boxed{\alpha < \frac{(m+M)g}{L}}$$

If $\frac{A}{B} < 0$ $(m+M)g - \alpha L < 0$

$$\boxed{\alpha > \frac{(m+M)g}{L}}$$

If $\frac{A}{B} = 0$, $(m+M)g - \alpha L = 0$

$$\boxed{\alpha = \frac{(m+M)g}{L}}$$

c) $B\{s^2\Theta(s) - s\theta(0^-) - \dot{\theta}(0^-)\} = (m+M)g\,\theta(s) - \alpha L\theta(s) - \beta L\,(s\theta(s) - \theta(0^-))$

$$\Theta(s)\left[Bs^2 - (m+M)g + \alpha L + \beta L s\right] = sB\,\theta(0^-) + B\dot{\theta}(0^-) + \beta L\,\theta(0^-)$$

$$\Theta(s) = \frac{sB\theta(0^-) + B\dot{\theta}(0^-) + \beta L\theta(0^-)}{Bs^2 + \beta L s - (m+M)g + \alpha L}$$

Poles at $\dfrac{-\beta L \pm \sqrt{\beta^2L^2 - 4B(\alpha L - (m+M)g)}}{2B}$

When $\beta \leq 0$, poles are not both in left-half plane $\Rightarrow$ unstable

When $\beta > 0$, poles are both in left-half plane if

$$\beta^2L^2 - 4B(\alpha L - (m+M)g) < \beta^2L^2$$

$$4B(\alpha L - (m+M)g) > 0$$

$$\alpha L - (m+M)g > 0$$

$$\alpha > \frac{(m+M)g}{L}$$

So broom will be balanced when $\boxed{\alpha > \frac{(m+M)g}{L} \quad \text{and} \quad \beta > 0}$

③ c) $G(s) = \frac{1}{2}$, $H_y(s) = e^{-s/3}$

$$\frac{Y(s)}{R(s)} = \frac{\frac{1}{2}}{1 + \frac{1}{2}e^{-s/3}} = \boxed{\frac{1}{2 + e^{-s/3}}}$$

To find impulse response, we'll use a different strategy than part a and b:

$$\frac{Y(s)}{R(s)} = \frac{1}{2 + e^{-s/3}}, \text{ rearrange:}$$

$$2Y(s) + e^{-s/3}Y(s) = R(s)$$

$$\Updownarrow \mathcal{L}^{-1}$$

$$2y(t) + y(t - \tfrac{1}{3}) = r(t)$$

$$2y(t) = r(t) - y(t - \tfrac{1}{3})$$

$$\Rightarrow y(t) = \tfrac{1}{2}\left(r(t) - y(t - \tfrac{1}{3})\right)$$

Plug in $\delta(t) = r(t)$, assuming $y(t<0) = 0$,

$$y(t) = \tfrac{1}{2}\left(\delta(t) - y(t - \tfrac{1}{3})\right)$$

$$y(0) = \tfrac{1}{2}\left(\delta(0) - \cancel{y(-\tfrac{1}{3})}^{\,0}\right)$$

y has δ @ $t=0$ w/ scale $\frac{1}{2}$

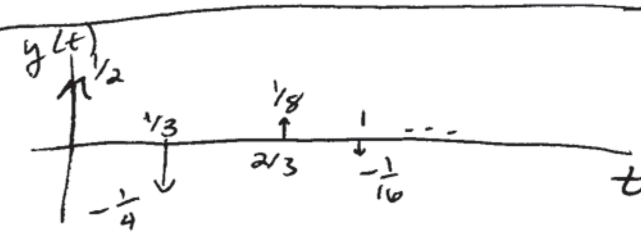
y has another value @ $t = \frac{1}{3}$

$$y(\tfrac{1}{3}) = \tfrac{1}{2}\left(\cancel{\delta(\tfrac{1}{3})}^{\,0} - y(0)\right)$$

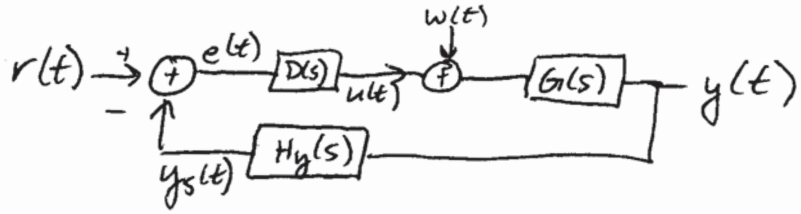
$$y(\tfrac{1}{3}) = -\tfrac{1}{2}\left(\tfrac{1}{2}\delta(0)\right) = -\tfrac{1}{4}\delta(t - \tfrac{1}{3})$$

$\cancel{\therefore\ y(t) = \sum_{n=0}^{\infty} (-1)^n}$

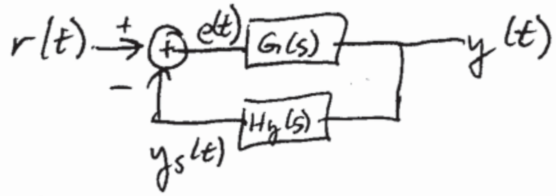
$$\therefore \boxed{y(t) = \sum_{n=0}^{\infty} (-1)^n \left(\tfrac{1}{2}\right)^{n+1} \delta\left(t - \tfrac{n}{3}\right)}$$



③



$w(t) = 0, \; D(s) = 1$



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H_y(s)}$$

a) $$\frac{Y(s)}{R(s)} = \left(\frac{1}{(s+1)(s+4)}\right)\frac{1}{1 + \frac{1}{(s+1)(s+4)}} = \frac{1}{(s+1)(s+4)+1}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{1}{s^2+5s+5}}$$

has Real poles, s_{p1}, s_{p2}

$$\frac{Y(s)}{R(s)} = \frac{1}{(s-s_{p1})(s\cancel{+}s_{p2})} = \frac{A}{s-s_{p1}} + \frac{B}{s-s_{p2}}$$

$$\frac{1}{s_{p1}-s_{p2}} = A \qquad \frac{1}{s_{p2}-s_{p1}} = B = -A$$

$$\frac{Y(s)}{R(s)} = \frac{\left(\frac{1}{s_{p1}-s_{p2}}\right)}{s-s_{p1}} - \frac{\left(\frac{1}{s_{p1}-s_{p2}}\right)}{s-s_{p2}}$$

$$h(t) = \frac{1}{s_{p1}-s_{p2}} e^{-s_{p1}t} u(t) - \frac{1}{s_{p1}-s_{p2}} e^{-s_{p2}t} u(t)$$

$$s_{p1,2} = \frac{-5 \pm \sqrt{25-20}}{2} = \frac{-5+\sqrt{5}}{2}, \; \frac{-5-\sqrt{5}}{2}$$

$$s_{p1} - s_{p2} = \sqrt{5}$$

$$\boxed{h(t) = \left(\frac{1}{\sqrt{5}} e^{\left(\frac{5-\sqrt{5}}{2}\right)t} - \frac{1}{\sqrt{5}} e^{\left(\frac{5+\sqrt{5}}{2}\right)t}\right) u(t)}$$

b) $$\frac{Y(s)}{R(s)} = \frac{\frac{1}{s+4}}{1 + \left(\frac{1}{s+1}\right)\left(\frac{1}{s+4}\right)} = \boxed{\frac{s+1}{s^2+5s+5}}$$

$$\frac{Y(s)}{R(s)} = \frac{s}{s^2+5s+5} + \frac{1}{s^2+5s+5}$$

$$h(t) = \frac{d}{dt}\left[\frac{1}{\sqrt{5}}\left(e^{\frac{5-\sqrt{5}}{2}t} - e^{\frac{5+\sqrt{5}}{2}t}\right)u(t)\right] + \frac{1}{\sqrt{5}}\left[e^{\frac{5-\sqrt{5}}{2}t} - e^{\left(\frac{5+\sqrt{5}}{2}\right)t}\right)u(t)\Big]$$

$$h(t) = \frac{1}{\sqrt{5}}\left[\cancel{\delta(t)} - \cancel{\delta(t)} + \frac{5-\sqrt{5}}{2} e^{\frac{5-\sqrt{5}}{2}t} - \frac{5+\sqrt{5}}{2} e^{\frac{5+\sqrt{5}}{2}t} + e^{\frac{5-\sqrt{5}}{2}t} - e^{\frac{5+\sqrt{5}}{2}t}\right] u(t)$$

$$\boxed{h(t) = \frac{1}{\sqrt{5}} u(t)\left[\frac{7-\sqrt{5}}{2} e^{\frac{5-\sqrt{5}}{2}t} - \frac{7+\sqrt{5}}{2} e^{\frac{5+\sqrt{5}}{2}t}\right]}$$

(c) a)

$$G(s) = \frac{\alpha}{s+\alpha}, \quad D(s) = K$$

$$\frac{Y(s)}{R(s)} = \frac{K\alpha}{s+\alpha}\left(\frac{1}{1+\frac{K\alpha}{s+\alpha}}\right) = \frac{K\alpha}{s+\alpha+\alpha K}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{K\alpha}{s+\alpha(1+K)}}$$

system is stable if poles in LHP.

$$\sigma_p + \alpha(1+K) = 0$$

$$\sigma_p = \underbrace{-\alpha(1+K) < 0}_{}$$

Implies:

for $\alpha > 0$, $-(1+K) < 0$

$1+K > 0$

$\boxed{K > -1}$

for $\alpha < 0$, $(1+K) < 0$

$\boxed{K < -1}$

$$E(s) = R(s) - \frac{K\alpha}{s+\alpha(1+K)} R(s)$$

plug in $u(t)$ for $r(t)$,

$$E(s) = \frac{1}{s} - \frac{K\alpha}{s+\alpha(1+K)}\frac{1}{s}$$

Final Val. Thm:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0}\left(1 - \frac{K\alpha}{s+\alpha(1+K)}\right)$$

$$= 1 - \frac{K\cancel{\alpha}}{\cancel{\alpha}(1+K)}$$

$$= \frac{1+K-K}{1+K} = \frac{1}{1+K}$$

$$\boxed{e(t) \rightarrow \frac{1}{1+K}} \neq 0.$$

(4) b) $G(s) = \frac{\alpha}{s+\alpha}$

$D(s) = K_1 + \frac{K_2}{s}$

$$\frac{Y(s)}{R(s)} = \frac{\alpha}{s+\alpha}\left(K_1 + \frac{K_2}{s}\right)\left(\frac{1}{1+\frac{\alpha}{s+\alpha}\left(K_1+\frac{K_2}{s}\right)}\right)$$

$$\frac{Y(s)}{R(s)} = \frac{\alpha K_1 + \alpha \frac{K_2}{s}}{s+\alpha+\alpha\left(K_1+\frac{K_2}{s}\right)}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{\alpha K_1 s + \alpha K_2}{s^2 + \alpha s + \alpha(K_1 s + K_2)}}$$

Stable if poles in LHP

Let $s_p = \dfrac{-\alpha(1+K_1) \pm \sqrt{\alpha^2(1+K_1)^2 - 4\alpha K_2}}{2}$

poles of this system

if s_p real, negative, then system is stable.

Need to show:

$$-\alpha(1+K_1) \pm \sqrt{\alpha^2(1+K_1)^2 - 4\alpha K_2} < 0$$

Case 1: $\alpha > 0$

$$-\alpha(1+K_1) + \sqrt{\alpha^2(1+K_1)^2 - 4\alpha K_2} < 0$$

$$+\alpha(1+K_1) > \sqrt{\alpha^2(1+K_1)^2 + -4\alpha K_2} \quad (>0)$$

$$\alpha^2(1+K_1)^2 > \alpha^2(1+K_1)^2 - 4\alpha K_2 > 0$$

$$0 > -4\alpha K_2 > -\alpha^2(1+K_1)^2$$

$$0 > -4K_2 > -\alpha(1+K_1)^2$$

$$\boxed{0 < K_2 < \alpha\frac{(1+K_1)^2}{4}}$$ K_2, K_1 positive

$$-\alpha(1+K_1) - \sqrt{\alpha^2(1+K_1)^2 - 4\alpha K_2} < 0$$ ✓

$$\cancel{\alpha(1+K_1) > \sqrt{\alpha^2(1+K_1)^2 < 4\alpha K_2}}$$

Case 2 $\alpha < 0 \rightarrow$ same

$$E(s) = R(s) - \left(\frac{Y(s)}{R(s)}\right) R(s)$$

plug in $r(t) = u(t)$, $R(s) = \frac{1}{s}$

$$E(s) = \frac{1}{s} - \frac{1}{s}\left(\frac{\alpha K_1 s + \alpha K_2}{s^2 + \alpha(1+K_1)s + \alpha K_2}\right)$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\left(1 - \frac{\alpha K_1 s + \alpha K_2}{s^2+\alpha(1+K_1)s + \alpha K_2}\right)$$

$$\boxed{\lim_{t\to\infty} e(t) = 1 - \frac{\alpha K_2}{\alpha K_2} = 0}$$

(4) c) $G(s) = \dfrac{1}{(s-1)^2}$

$D(s) = K_1 + \dfrac{K_2}{s} + K_3 s$

$$\frac{Y(s)}{R(s)} = \frac{1}{(s-1)^2}\left(K_1 + \frac{K_2}{s} + K_3 s\right)\frac{1}{1 + \frac{1}{(s-1)^2}\left(K_1 + \frac{K_2}{s} + K_3 s\right)}$$

$$\frac{Y(s)}{R(s)} = \frac{K_1 + \frac{K_2}{s} + K_3 s}{(s-1)^2 + K_1 + \frac{K_2}{s} + K_3 s}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{K_1 s + K_2 + K_3 s^2}{s(s-1)^2 + K_1 s + K_2 + K_3 s^2}}$$

System is stable if all poles in LHP.
Sufficient to show that any poles can be chosen with real K_1, K_2, K_3

We will show that $s_p = -1, -2, -3$ yields a particular solution for K_1, K_2, K_3.

Denominator:

$$s^3 + (K_3 - 2)s^2 + (K_1 + 1)s + K_2 = 0$$

Let $s = -1$:

$$-1 + K_3(1) - 2 + K_1(-1) - 1 + K_2 = 0$$

(*) $K_3 - K_1 + K_2 = 1 + 2 + 1 = 4$

Let $s = -2$,

(**) $4K_3 - 2K_1 + K_2 = 18$

Let $s = -3$

(***) $9K_3 - 3K_1 + K_2 = 48$

$4(*) - (**) = \boxed{-2K_1 + 3K_2 = -2}$

$9(*) - (***) = \boxed{-6K_1 + 8K_2 = -12}$

$$\Rightarrow \boxed{\begin{aligned} K_1 &= 10 \\ K_2 &= 6 \\ K_3 &= 8 \end{aligned}}$$

4 c) cont'd

Use Final Value thm to get $e(t) \to 0$

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$E(s) = R(s) - \frac{Y(s)}{R(s)} = \frac{1}{s} - \frac{1}{s}\left(\frac{K_1 s + K_2 + K_3 s^2}{s(s-1)^2 + K_1 s + K_2 + K_3 s^2}\right)$$

$u(t) \Rightarrow \frac{1}{s}$

$$\lim_{s \to 0} sE(s) = \lim_{s \to 0}\left(1 - \frac{K_1 s + K_2 + K_3 s^2}{s(s-1)^2 + K_1 s + K_2 + K_3 s^2}\right)$$

$$\lim_{s \to 0} sE(s) = 1 - \frac{K_2}{0 + K_2} = 1 - 1 = 0$$

$\therefore$ $\boxed{e(t) \to 0}$

If we have P-I controller,

$K_3 = 0$,

so the denominator is:

$$s^3 - 2s^2 + (K_1 + 1)s + K_2$$

~~this has roots when~~

~~$s^3 + (K_1 + 1)s + K_2 = 2s^2$~~

The negative sign on $2s^2$ means that we will always have ~~at~~ at least 1 root with real component > 0, RHP.

5) i) a) $\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$\frac{Y(s)}{R(s)} = \frac{\frac{10s+1}{s^2+s+1}}{1+\frac{10s+1}{s^2+s+1}} = \boxed{\frac{10s+1}{s^2+11s+2}}$$

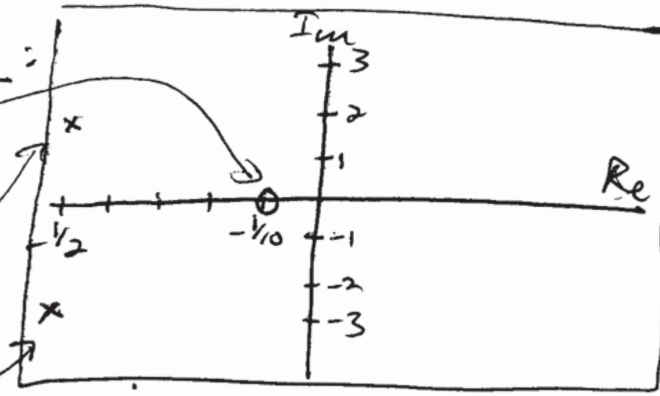
i) b) Pole Zero:

$z_1 = -\frac{1}{10}$

$P_{1,2} = \frac{-1 \pm \sqrt{-3}}{2}$

$P_1 = -\frac{1}{2} + i\sqrt{3}$

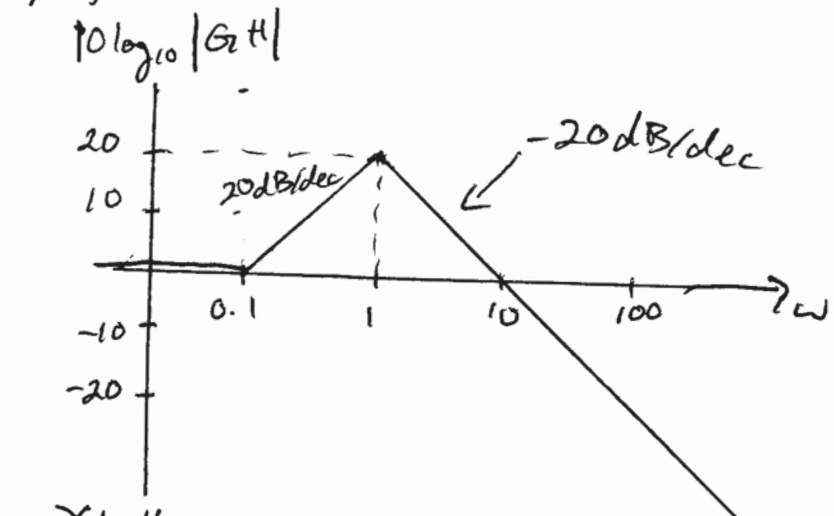
$P_2 = -\frac{1}{2} - i\sqrt{3}$

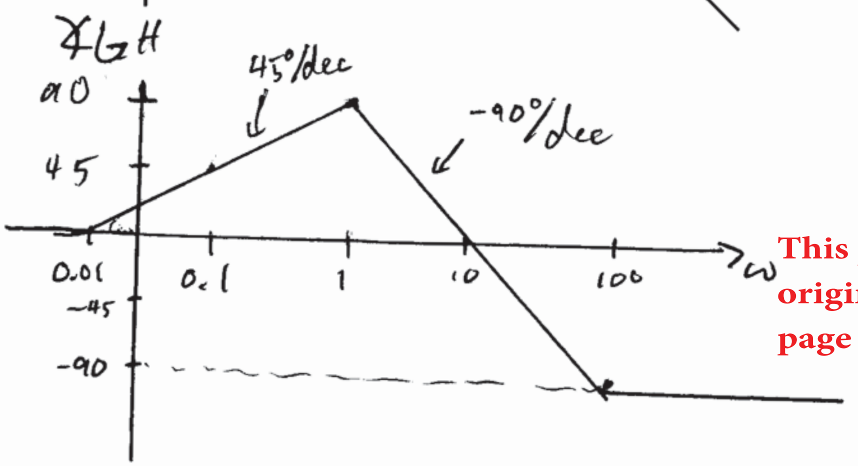


Phase Margin ~ 180°

Gain Margin ~ ∞

i) c)

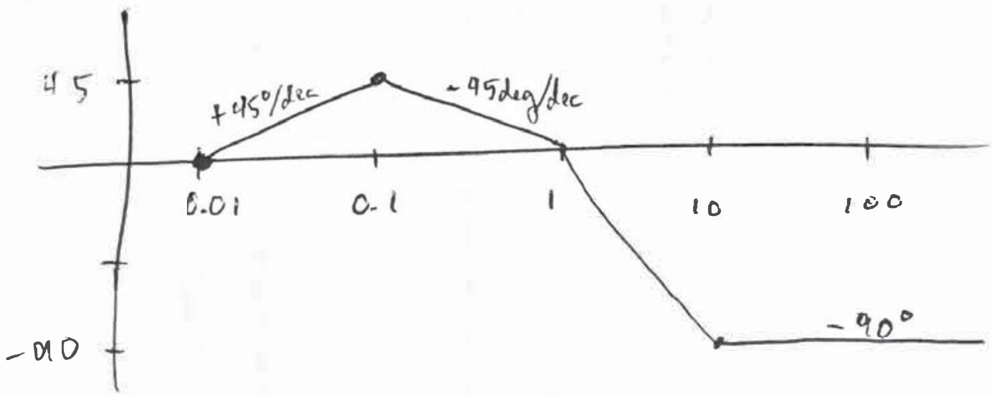




This phase plot is incorrect in original solution. Please see next page for correct phase plot.

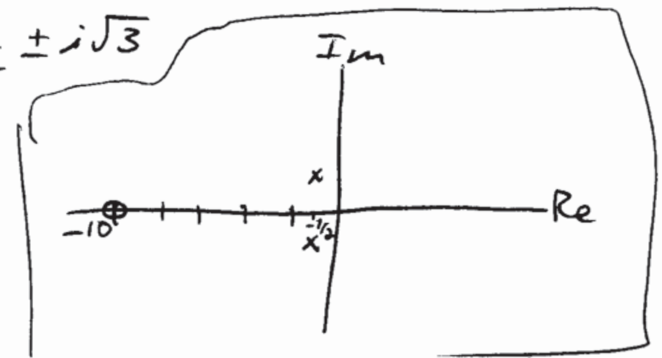
PS10 problem 5 a) **Correct Phase plot for 5-c-i**

i) Corrected phase plot
$\measuredangle G(f)$



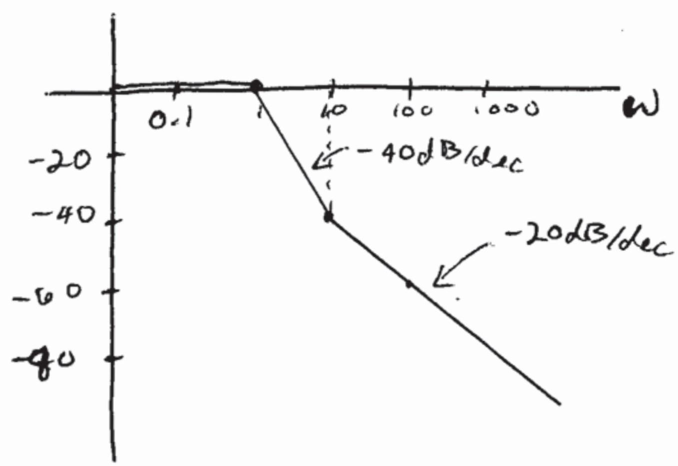
5 ii) a) $\dfrac{Y(s)}{R(s)} = \dfrac{s/10+1}{s^2+s+1}\left(\dfrac{1}{1+\dfrac{s/10+1}{s^2+s+1}}\right) = \dfrac{s/10+1}{s^2+s+1+s/10+1} = \boxed{\dfrac{s/10+1}{s^2+\frac{11s}{10}+2}}$

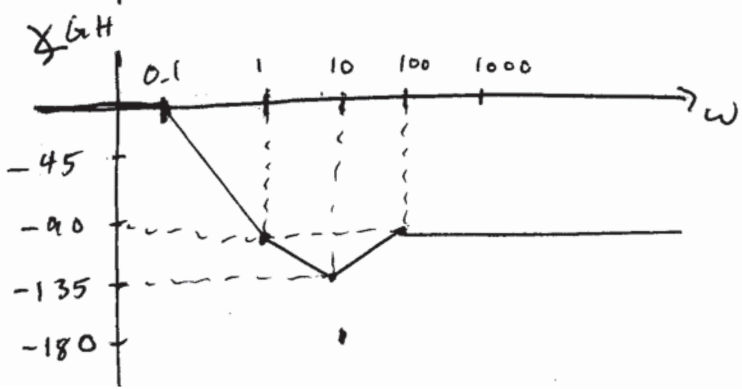
b) GH pole-zeros: $z_1 = -10$ $\quad P_{1,2} = -\frac{1}{2} \pm i\sqrt{3}$



ii)

$10\log_{10}|GH|^2$



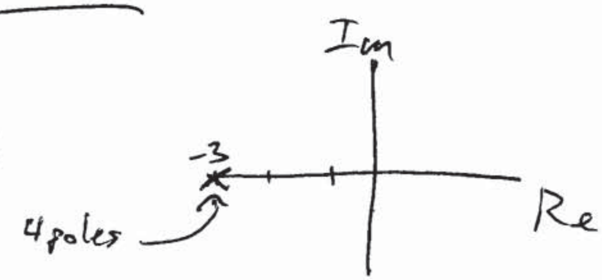


Phase Margin ~ 90°

Gain Margin ~ ∞

5 iii) a) $\frac{Y(s)}{R(s)} = \frac{\frac{1}{(s+3)^3}}{1+\frac{1}{(s+3)^3(s+3)}} = \frac{1}{(s+3)^3 + \frac{1}{s+3}} = \boxed{\frac{s+3}{(s+3)^4+1}}$

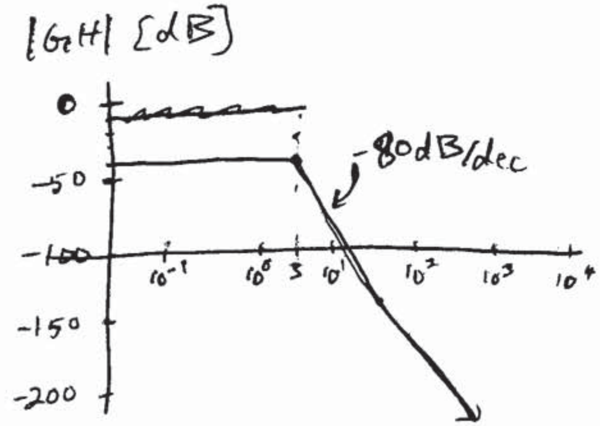
b) 4 poles for $GH = \frac{1}{(s+3)^4}$, all @ $s = -3$

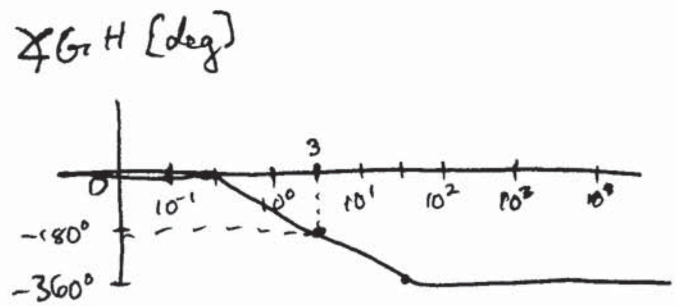


c) $GH(j\omega) = \frac{1}{(j\omega+3)^4} = \frac{\left(\frac{1}{3}\right)^4}{\left(1+\frac{j\omega}{3}\right)^4}$

$K = \left(\frac{1}{3}\right)^4 \rightarrow$

$w_c = 3$



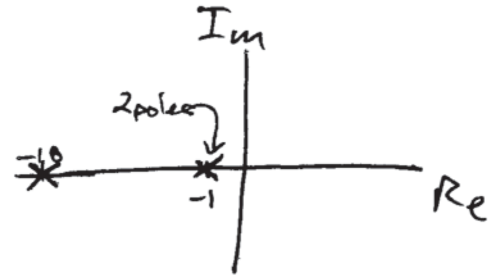


d) $\boxed{\text{Gain margin} \sim +40\,dB}$

$\boxed{\text{Phase margin} \sim \infty^\circ}$

5 iv) a) $\frac{Y(s)}{R(s)} = \frac{1}{(s+1)^2(s+10)}\left(\frac{1}{1+100/(s+1)^2(s+10)}\right) = \boxed{\frac{1}{(s+1)^2(s+10)+100}}$

b) $G(s)H_y(s) = \frac{100}{(s+1)^2(s+10)}$: 2 poles @ $s=-1$
1 pole @ $s=-10$

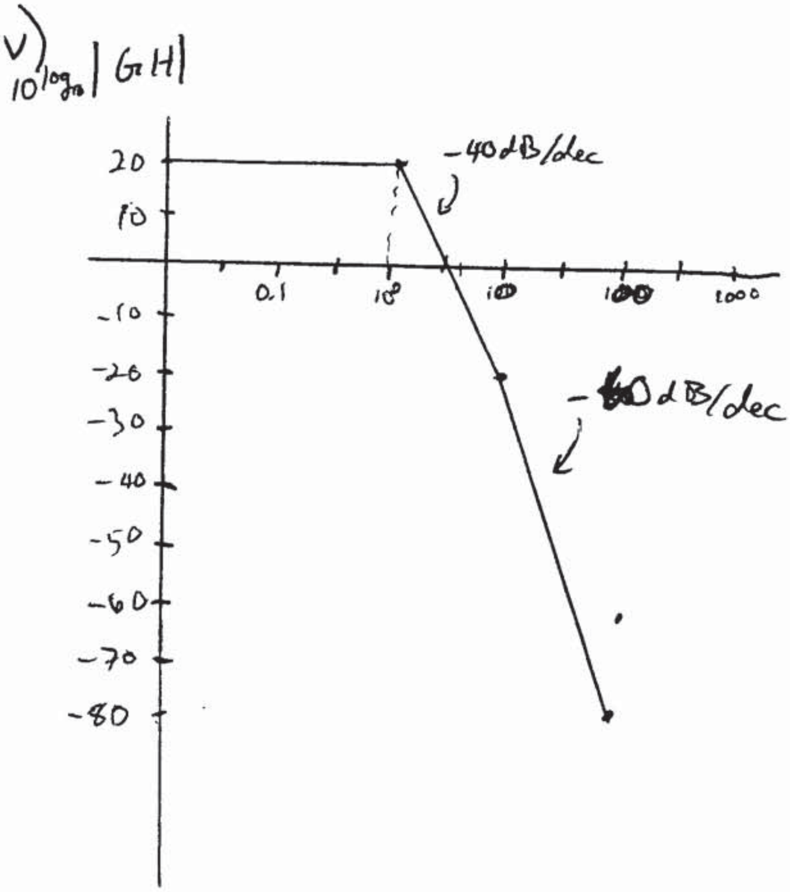


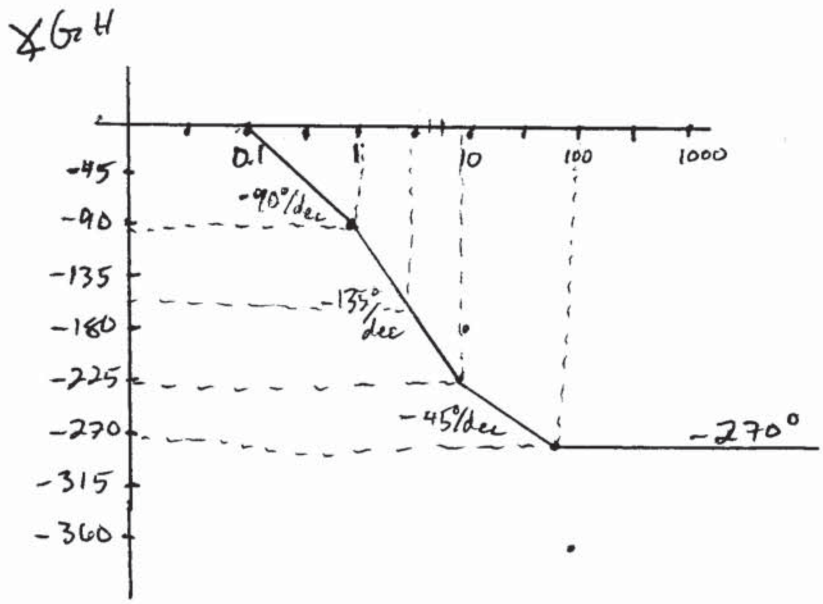
c) $GH_y(j\omega) = \frac{100\,\frac{1}{10}}{(j\omega+1)^2(\frac{j\omega}{10}+1)} = \frac{10}{(1+\frac{j\omega}{1})^2(\frac{j\omega}{10}+1)}$

$K=10$

$\omega_c=1,\ N=2$ $\qquad$ $\omega_c=10,\ N=1$

iv)





By inspection

Phase margin ~ 20°

Gain margin: ~6 dB