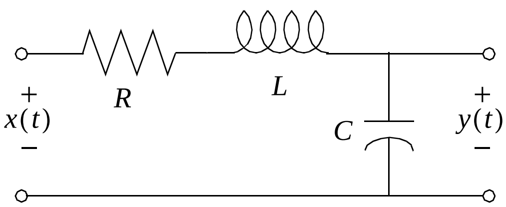
University of California at Berkeley
College of Engineering
Department of Electrical Engineering and Computer Sciences

EECS 120: Signals and Systems
Fall Semester 1998

Examples of Simple LTI Systems

1. Continuous-Time, Second-Order Lowpass Filter



This circuit is governed by the differential equation:

$$\frac{d^2y}{dt^2} + \frac{R}{L}\frac{dy}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t) .$$

We can transform the differential equation into a standard form by defining the *natural frequency* $\omega_n = 1/\sqrt{LC}$ and the *damping constant* $\zeta = (R/2)\sqrt{C/L}$:

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t) .$$

The system exhibits three regimes of behavior:

Underdamped	$0 < \zeta < 1$
Critically damped	$\zeta = 1$
Overdamped	$\zeta > 1$

Impulse Response

Overdamped: if $\zeta > 1$, characteristic equation has two distinct, real roots, and:

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left[e^{-\omega_n(\zeta - \sqrt{\zeta^2 - 1})t} - e^{-\omega_n(\zeta + \sqrt{\zeta^2 - 1})t}\right]u(t) .$$

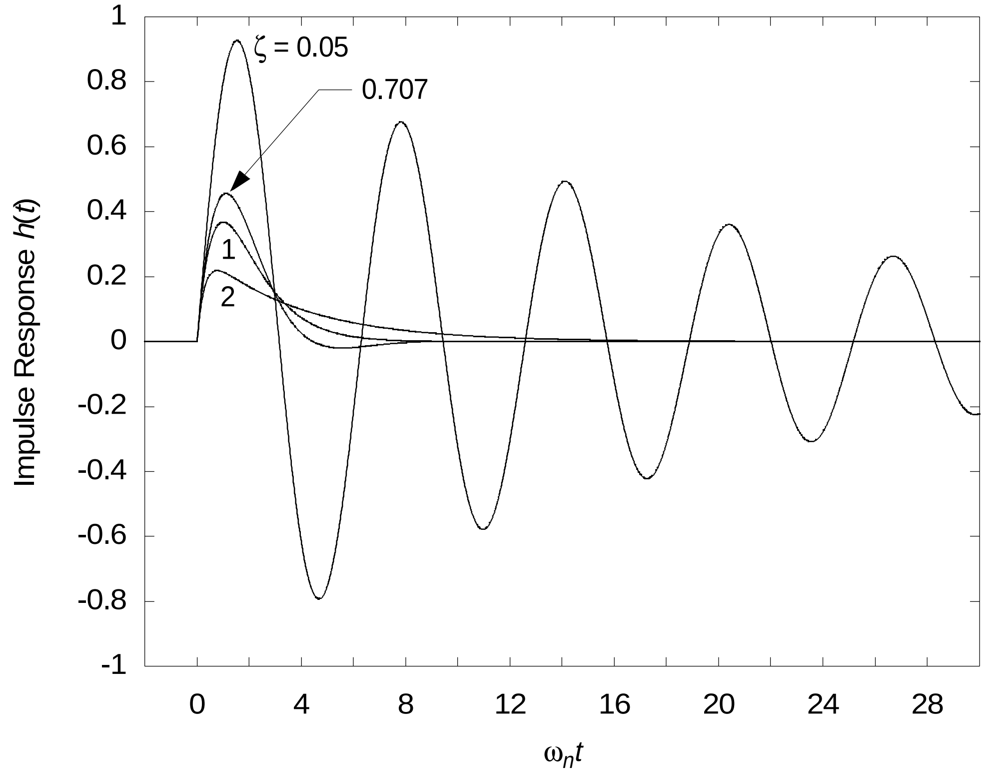
An error in the overall sign of this equation was corrected

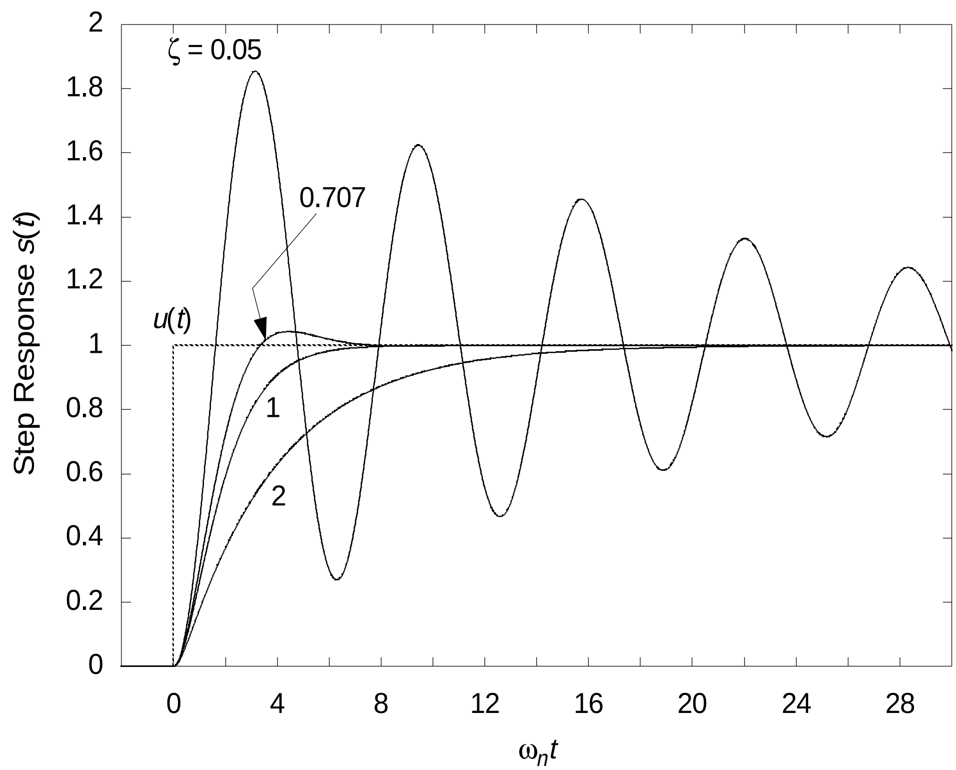
Underdamped: if $0 \le \zeta < 1$, characteristic equation has two distinct, complex conjugate roots, and:

$$h(t) = \frac{\omega_n}{2j\sqrt{1 - \zeta^2}}e^{-\zeta\omega_n t}\left(e^{j\sqrt{1-\zeta^2}\omega_n t} - e^{-j\sqrt{1-\zeta^2}\omega_n t}\right)u(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}}e^{-\zeta\omega_n t}\sin\left(\sqrt{1 - \zeta^2}\,\omega_n t\right)u(t) .$$

Critically Damped: if $\zeta = 1$, characteristic equation has one real root of multiplicity two, and:

$$h(t) = \omega_n^2 t e^{-\omega_n t}u(t) .$$

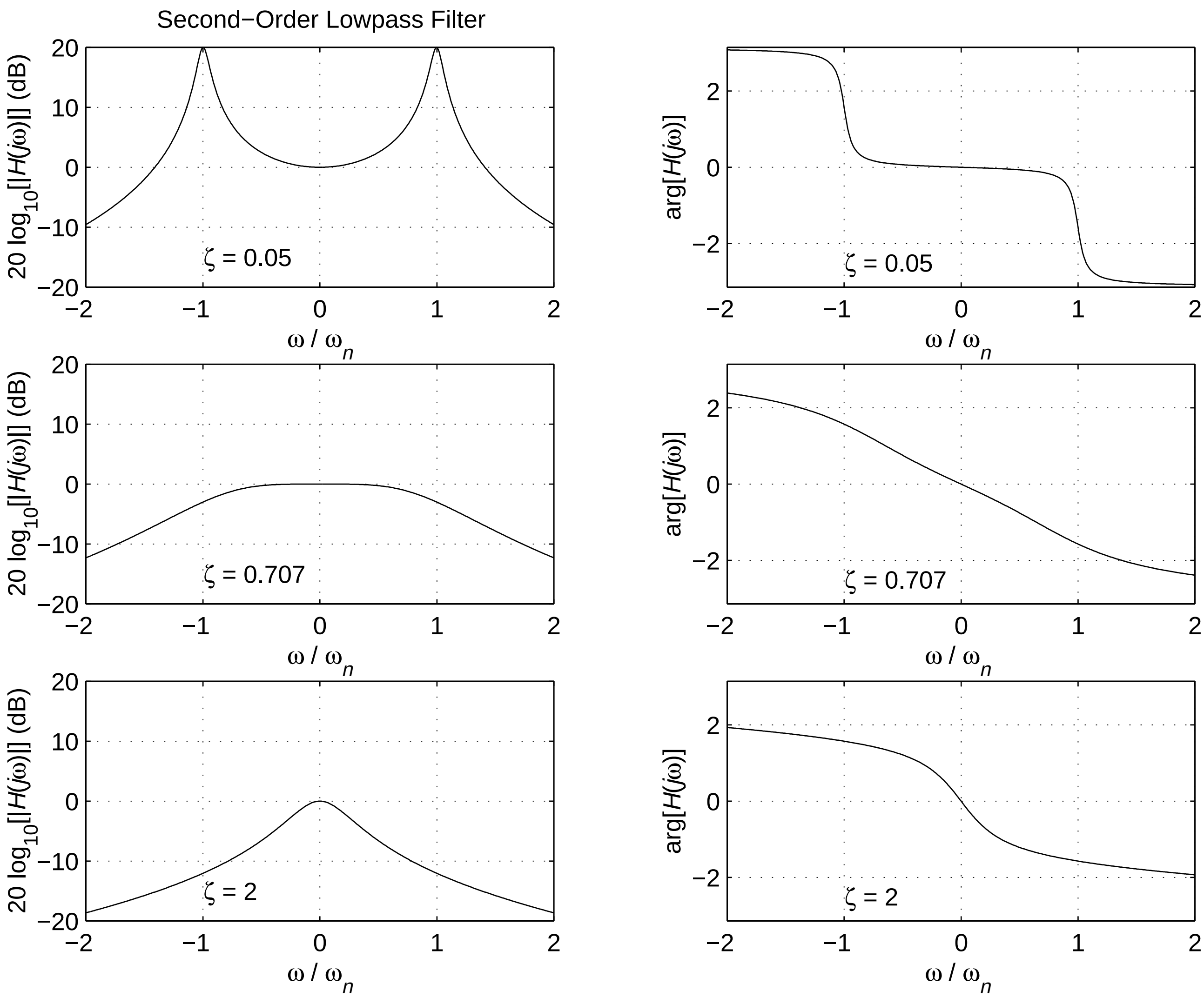

ζ = 0.05
0.707
1
2
Impulse Response h(t)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
4
8
12
16
20
24
28
ωnt


ζ = 0.05
0.707
u(t)
1
2
Step Response s(t)
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
4
8
12
16
20
24
28
ωnt

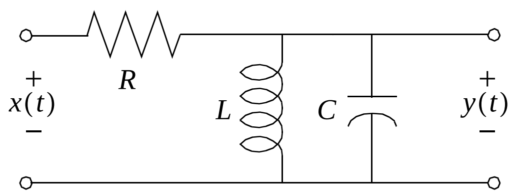
Frequency Response

As long as $\zeta > 0$, the frequency response $H(j\omega)$ exists and is given by:

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} .$$



2. Continuous-Time, Second-Order Bandpass Filter



This circuit is governed by the differential equation:

$$\frac{d^2y}{dt^2} + \frac{1}{RC}\frac{dy}{dt} + \frac{1}{LC}y(t) = \frac{1}{RC}\frac{dx}{dt}.$$

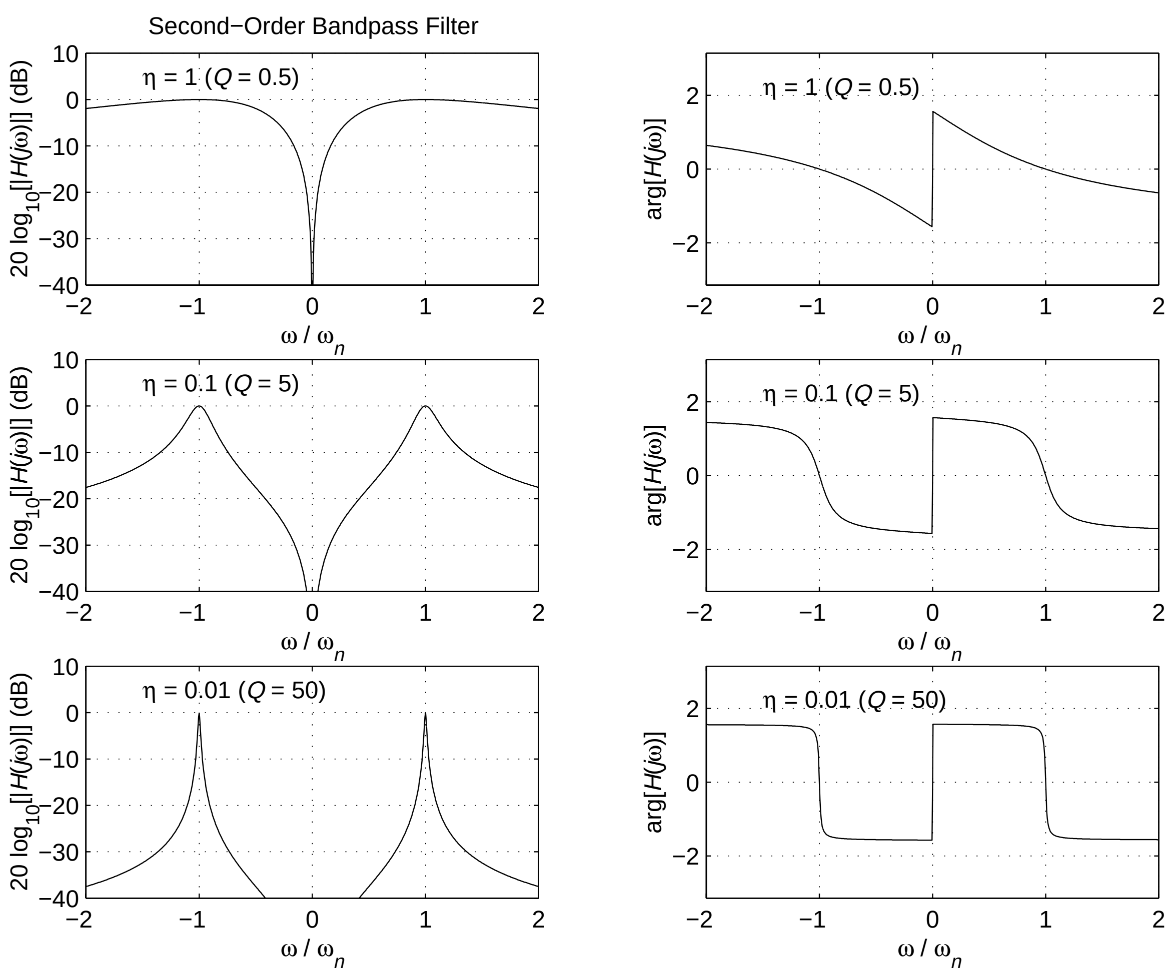
Defining $\omega_n = 1/\sqrt{LC}$ and $\eta = (1/2R)\sqrt{L/C}$:

$$\frac{d^2y}{dt^2} + 2\eta\omega_n\frac{dy}{dt} + \omega_n^2 y(t) = 2\eta\omega_n\frac{dx}{dt}.$$

The frequency response $H(j\omega)$ is given by:

$$H(j\omega) = \frac{2\eta\omega_n(j\omega)}{(j\omega)^2 + 2\eta\omega_n(j\omega) + \omega_n^2}.$$

(In circuit analysis, it is customary to define the *quality factor*: $Q = \omega_n RC = \frac{1}{2\eta}$).



3. Causal, FIR Approximation to Discrete-Time, Ideal Lowpass Filter

Ideal Lowpass Filter

Frequency Response: $H(e^{j\Omega}) = \begin{cases} 1, & |\Omega| \le W \\ 0, & W < |\Omega| \le \pi \end{cases}$, where $W < \pi$, and $H(e^{j(\Omega + 2\pi)}) = H(e^{j\Omega})$.

Impulse Response (noncausal, IIR): $h[n] = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wn}{\pi}\right)$.

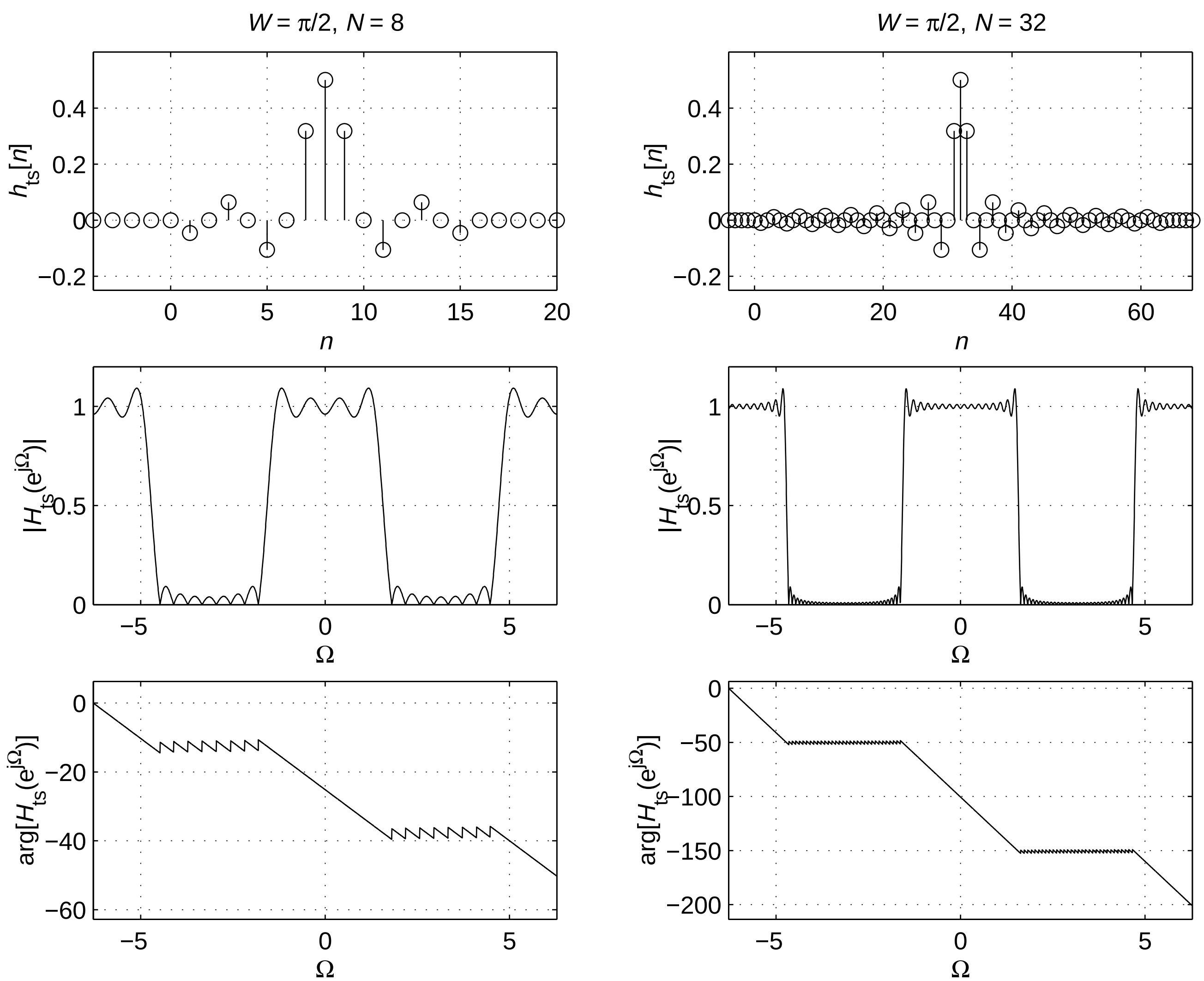
Causal, FIR Approximation

Truncated Impulse Response (noncausal, FIR): $h_{\mathrm{trunc}}[n] = \begin{cases} \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wn}{\pi}\right), & |n| \le N \\ 0, & |n| > N \end{cases}$.

Shifted, Truncated Impulse Response (causal, FIR):

$$h_{\mathrm{trunc,shift}}[n] = h_{\mathrm{trunc}}[n-N] = \begin{cases} \frac{W}{\pi}\mathrm{sinc}\left(\frac{W(n-N)}{\pi}\right), & 0 \le n \le 2N \\ 0, & n < 0,\ \mathrm{n} > 2N \end{cases}.$$

Frequency Response: $H_{\mathrm{trunc,shift}}(e^{j\Omega}) = \frac{W}{\pi}\sum_{n=0}^{2N}\mathrm{sinc}\left(\frac{W(n-N)}{\pi}\right)e^{-jn\Omega}$.



4. Simple Discrete-Time, Second-Order System

Difference Equation: $y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n]$, $r > 0$, $0 \le \theta \le \pi$.

Impulse Response (IIR):

Critically damped:

For $\theta = 0$, characteristic equation has one real root of multiplicity two, and: $h[n] = (n+1)r^n u[n]$.

Underdamped:

For $0 < \theta < \pi$, there are two distinct, complex conjugate roots, and: $h[n] = r^n \dfrac{\sin[(n+1)\theta]}{\sin\theta} u[n]$.

For $\theta = \pi$, there is one real root of multiplicity two, and: $h[n] = (n+1)(-r)^n u[n]$.

Frequency Response:

For $0 < r < 1$, $H(e^{j\Omega})$ exists and is given by: $H(e^{j\Omega}) = \dfrac{1}{1 - (2r\cos\theta)e^{-j\Omega} + r^2 e^{-j2\Omega}}$.