Chapter 3

Modulation

This chapter covers the modulation techniques for AM, FM, TV. These techniques shift a baseband signal into a bandpass signal (centered around 0) that fits in a transmission channel centered around a carrier frequency. The following table summarizes most modulation techniques. The table uses the fact (proved later) that since the modulated signal $x$ is a bandpass signal with carrier frequency $\omega_c$ rad/sec, $x$ can be represented in the form: $\forall t \in \text{Reals}$,

$$x(t) = \text{Re}\{A(t)e^{j(2\pi\omega_c t + \theta(t))}\}.$$  

The table below lists the name of the modulation scheme, and the corresponding signals $z(t) = A(t)e^{j\theta(t)}$, $A(t)$, $\theta(t)$. Also $m$ denotes the modulating or message signal.

<table>
<thead>
<tr>
<th>Modulation scheme</th>
<th>$z(t)$</th>
<th>$A(t)$</th>
<th>$\theta(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AM-LC</td>
<td>$A[1 + \beta m(t)]$</td>
<td>$A[1 + \beta m(t)]$</td>
<td>0</td>
</tr>
<tr>
<td>AM-DSB-SC</td>
<td>$Am(t)$</td>
<td>Am(t)</td>
<td>0</td>
</tr>
<tr>
<td>AM-SSB</td>
<td>$A[m(t) \pm j\hat{m}(t)]$</td>
<td>$A[m^2(t) + \hat{m}^2(t)]^{1/2}$</td>
<td>$\tan^{-1} \pm \hat{m}(t)$</td>
</tr>
<tr>
<td>VSB</td>
<td>$Ae^{j\beta m(t)}$</td>
<td>$A$</td>
<td>$\beta m(t)$</td>
</tr>
<tr>
<td>PM</td>
<td>$Ae^{j\phi_0 + \int_0^t \beta m(s) ds}$</td>
<td>$A$</td>
<td>$\phi_0 + \int_0^t \beta m(s) ds$</td>
</tr>
<tr>
<td>FM</td>
<td>$Ae^{j\phi_0 + \int_0^t \beta m(s) ds}$</td>
<td>$A$</td>
<td>$\phi_0 + \int_0^t \beta m(s) ds$</td>
</tr>
</tbody>
</table>

Table 3.1: Representation of different modulation schemes
1. What is modulation?

A modulator is a system:

- message signal: $m(t)$, $t \in \mathbb{R}$
- carrier signal: $e.g., \cos 2\pi f_c t$

Example 1 – (Shortwave broadcast) AM

$$x(t) = m(t) \cos [2\pi \times 3 \times 10^5 t]$$

\[
\begin{align*}
[H(m)](t) &= x(t) \\
\text{message signal} &\quad \text{modulated signal}
\end{align*}
\]

$H$ is usually time-varying; may be linear (AM) or non-linear (FM)

Example 1 – (Shortwave broadcast) AM
Example 2 – “Cheap” modem

2. Assumptions

A. \( m(t) \leftrightarrow M(f) \). Assume \( m \) is bandlimited to \( |f| < B_m \) Hz. Also, \( |m(t)| \leq 1 \), all \( t \).

\[
|M(f)|
\]

\(-B_m \quad B_m \quad f \text{ Hz}\)

B. \( x(t) \leftrightarrow X(f) \). Assume \( x \) is bandpass signal centered around \( f_c \) and with bandwidth \( B_x \).

\[
|X(f)|
\]

\(-f_c \quad f_c \quad f \text{ Hz}\)

C. \( B_x \ll f_c \). We say that \( x \) is a narrowband signal.

3. Why modulation?

A. Propagation. It may not be possible to transmit a baseband signal.
B. Channel sharing

C. Noise immunity: FM better than AM. Go to EECS 121.

4. Linear Modulation (Modulation is a linear system) (but not time invariant!)

A. Double side band AM (DSB-AM)

Objective

\[ |M(f)| \quad \xrightarrow{\text{Modulator}} \quad -f_c \rightarrow f \rightarrow f_c \]

Idea

\[ m(t) \cdot e^{j2\pi f_c t} \leftrightarrow \delta[m] * \delta[e^{j2\pi f_c t}] = M(f) * \delta(f-f_c) = M(f-f_c) \]

Better idea

\[ x(t) = m(t) \cos 2\pi f_c t \]

\[ \frac{1}{2} |M(f)| * \begin{cases} 
 1 & \text{for } f = -f_c, f_c \\
 0 & \text{elsewhere} 
\end{cases} = \frac{1}{2} |X(f)| \quad \text{for } f = -f_c, f_c \]

\[ X(f) = \frac{1}{2} \{ M(f-f_c) + M(f+f_c) \} \]

Note, \( B_x \ll f_c \).
Demodulation

Demodulation must have local oscillator (at $f_c$) where phase is “locked” to carrier. This requires a phase-locked loop. Suppose there is a phase difference:

Cheap Solution: DSB with large carrier (DSB-LC)

Suppose $|m(t)| \leq 1$. Let $\mu < 1 \ (\approx 0.8)$ modulation index. Let

$$x(t) = [1 + \mu m(t)] \cos(2\pi f_c t)$$

$$= \cos 2\pi f_c t + \mu m(t) \cos 2\pi f_c t$$
\[
\frac{1}{2} \mu M(f + f_c) \quad [X(f)] \\
\frac{1}{2} \mu M(f - f_c) \\
- f_c \\
f_c \\
\mu m(t) \\
x(t) \\
[1 + \mu m(t)] \\
t \\
y(t) \\
\text{envelope detector} \\
\text{Rectifier} \\
\text{LPF} \\
z(t) \\
x(t) \text{ is high frequency } (f_c) \text{ sinusoid whose "envelope" is } [1 + \mu m(t)].
\]

\[
z(t) \equiv [1 + \mu m(t)] \text{ is the "envelope."}
\]

**Exercise:** Determine \( Y(f), Z(f) \) and show the envelope detector works.

**Drawback:**

\[
P_x = \frac{1}{2} + \frac{1}{2} \mu^2 P_m
\]

\[
P_{\text{carrier}} \quad P_{\text{signal}}
\]

\[
\text{Efficiency} = \frac{P_{\text{signal}}}{P_x} = \frac{\mu^2 P_m}{1 + \mu^2 P_m} < 50\%
\]

**Difficulty B**

DSB and DSB-LC consume bandwidth \( B_x = 2B_m \), i.e., twice message bandwidth.

**Idea:** Make \( B_x = B_m \) by:

\[
[X_u(f)] \\
- f_c \sim f_c \\
\text{SSB modulation}
\]

OR

\[
[X_d(f)] \\
- f_c \sim f_c \\
\text{"upper" sidebands}
\]

\[
\text{"lower" sidebands}
\]

\[
\mu^2 P_m \sim B_x \sim B m
\]
Scheme 1

Scheme 1 requires filters with sharp cut-off.

Scheme 2 – Phase Shift Method

Idea – Review Hilbert Transform

\( m(t) \leftrightarrow M(f) \)

\( m(t) + j\hat{m}(t) \leftrightarrow M(f) + j\hat{M}(f) = 2M(f)u(f) \)

\( \{m(t) + j\hat{m}(t)\}e^{j2\pi f_c t} \leftrightarrow 2M(f-f_c)u(f-f_c) \)

\( \text{SSB-U Modulator} \)

\( x_u(t) = Re\{(m(t) + j\hat{m}(t))e^{j2\pi f_c t}\} 

= \text{Re}\{m(t)\cos 2\pi f_c t - \hat{m}(t)\sin 2\pi f_c t\} \)
Similarly:

\[ x_d(t) = m(t) \cos 2\pi f_c t + \dot{m}(t) \sin 2\pi f_c t \]

\[ \leftrightarrow \]

\[ -f_c \sim \sim f_c \]

SSB-U Demodulator

\[ m(t) \]

90° \( \phi \)-shifter

HTF

\(-j \text{sgn} f\)

\[ \hat{m}(t) \]

\[ \sin 2\pi f_c t \]

\[ \cos 2\pi f_c t \]

\[ \pm \]

\[ x_u(t) \]

SSB-U Demodulator

\[ x_u \]

\[ y(t) \]

LPF

\[ z(t) \]

\[ \cos 2\pi f_c t \]

\[ x_u(t) = m(t) \cos 2\pi f_c t - \dot{m}(t) \sin 2\pi f_c t \]

\[ y(t) = \frac{1}{2}m(t) + \frac{1}{2}m(t) \cos 4\pi f_c t - \frac{1}{2}\dot{m}(t) \sin 4\pi f_c t \]

\[ z(t) = \frac{1}{2}m(t) \]

AM Receiver: Superheterodyne

AM band: \( 540 - 1700 \text{ kHz} \)

10 kHz = channel width

AM Receiver:

In principle,
Difficulties:

1) large gain

2) sharp filters

Solution: \( \omega_c + 2 \omega_f \) (image) Let \( \omega_c = 2 \pi \) (1000 kHz)

Stereo-FM
Note:
- compatible
  (stereo on mono receiver ⇒ mono reception)
- reverse compatible
  (mono on stereo receiver ⇒ mono reception)

TV
BW: 485 lines in 2 fields + 2 × 20 line durations (vertical retrace)
× 30/sec. ⇒ line frequency: \( f_2 = 15750 \) lines/sec

Bandwidth:
\[
525 \times 525 \times \frac{4}{3} \times 30 \text{ points/sec.}
\]
worst case: BWBW ...
\[
\Rightarrow \quad \frac{1}{2} 525 \times 525 \times \frac{4}{3} \times 30 = 5.5 \text{ MHz}
\]
In practice: 4.2 MHz – too big to fit in 6MHz channel
VSB (Vestigial Side Band)

\[ y(t) = k \cos \omega_c t + \text{SSB} - g(t) \sin \omega_c t, \text{ why?} \]

\[ = (k + m(t)) \cos \omega_c t - (\dot{m}(t) + g(t)) \sin 2\pi f_0 \]

Envelope Detection: (with residual carrier)

Also: Sound = 10 kHz max.

\[ B \approx 80 \text{ kHz} \]

FM Modulation: \((z(t))\)

\[ 2(\omega) = \text{video + audio (BW)} \]
Exponential Modulation

\[ x(t) = A \cos[2\pi f_c t + \theta(t)] \]

A. Phase Modulation

\[ \theta(t) = \phi_\Delta \cdot m(t) \]
\[ x(t) = A \cos[2\pi f_c t + \phi_\Delta \cdot m(t)] \]
\[ |\phi_\Delta| \leq \pi \]

B. Frequency Modulation

\[ \theta(t) = 2\pi f_\Delta \int_0^t m(s) ds \]
\[ x(t) = A \cos \left[ 2\pi f_c t + 2\pi f_\Delta \int_0^t m(s) ds \right] \]

Narrowband PM/FM

\[ x(t) = \cos[2\pi f_c t + \theta(t)] = Re\{e^{j[2\pi f_c t + \theta(t)]}\} \]

Assume \( |\theta(t)| \ll 1 \), so

\[ e^{j\theta(t)} = 1 + j\theta(t) + \frac{j^2}{2!} \theta^2(t) + \frac{j^3}{3!} \theta^3(t) + \ldots \]
\[ \equiv 1 + j\theta(t) \]

So

\[ x(t) \equiv Re\{[1 + j\theta(t)]e^{j2\pi f_c t}\} \]
\[ = \cos 2\pi f_c t - \theta(t) \sin 2\pi f_c t \]
NB-PM: \( \theta(t) = \phi_m m(t) \). So,

\[
x(t) = \cos 2\pi f_c t - \phi_m m(t) \sin 2\pi f_c t
\]

\[
X(f) = \frac{1}{2} \left( \delta(f-f_c) + \delta(f+f_c) \right) - \phi_m \cdot \frac{1}{2j} \left\{ M(f-f_c) - M(f+f_c) \right\}
\]

\[
= \frac{1}{2} \left( \delta(f-f_c) + \delta(f+f_c) \right) + \frac{1}{2} \phi_m \cdot j \left\{ M(f-f_c) - M(f+f_c) \right\}
\]

Time domain comparison between DSB-LC and narrowband PM:

[Amplitude of phasor in NB-PM case is \( (1 + \phi_m^2 m(t)^2)^{1/2} = (1 + \theta^2(t))^{1/2} = 1 \! ]

\[
\text{NB-FM}
\]

\[
|\theta(t)| < 1
\]

\[
x(t) = \cos \left[ 2\pi f_c t + \theta(t) \right] \equiv \cos 2\pi f_c t - \theta(t) \sin 2\pi f_c t
\]

\[
\theta(t) = 2\pi f_c \cdot \int m(s) ds
\]

\[
\left| \theta(t) \right| \ll 1 \Rightarrow m \text{ has no dc component, } M(f)\big|_{f=0} = 0
\]

\[
(1) \Rightarrow X(f) = \frac{1}{2} \left( \delta(f-f_c) + \delta(f+f_c) \right) - \frac{1}{2j} \left\{ \Theta(f-f_c) - \Theta(f+f_c) \right\}
\]

\[
(2) \Rightarrow \Theta(f) = 2\pi f_c \cdot \frac{M(f)}{j2\pi f} = -j f_c \cdot \frac{M(f)}{f}
\]
So, \[ X(f) = \frac{1}{2} \left( \delta(f - f_c) + \delta(f + f_c) \right) + \frac{f_A}{2} \left( \frac{M(f - f_c)}{f - f_c} - \frac{M(f + f_c)}{f + f_c} \right) \]

|\[M(f)| = \alpha |f| \]
\[
\begin{array}{c}
\frac{1}{2} f_A \alpha/2 \quad |\alpha(t)|
\end{array}
\]

\[
\begin{array}{c}
\frac{1}{2} f_A \alpha/2 \quad |\alpha(t)|
\end{array}
\]

**NB-Modulator**

\[ x(t) = \cos 2\pi f_c t - \theta(t) \sin 2\pi f_c t \]

\[
m(t) \quad \phi_{\Delta} \quad \theta(t) \quad m(t) \quad f_A \int_0^t \quad \theta(t)
\]

PM

FM

**NB-FM-Demodulator**

First, do representation of bandpass signals.

**Definition:** \( x(t), \quad t \in \mathbb{R} \), is a real bandpass signal with carrier \( f_c \) if \( |X(f)| = 0 \) for \( |f - f_c| > \frac{1}{2} B_x \) and \( B_x \ll f_c \):

\[
\begin{array}{c}
- f_c \\
\sim \\
0 \\
\sim \\
f_c \\
\hline
\end{array}
\]

\[
|X(f)|
\]

**Theorem:** Let \( x \) be a real bandpass signal with carrier \( f_c \). Let \( \hat{x} = \text{HT} \) of \( x \). Then \( x, \hat{x} \) have the representation

\[
x(t) = A(t) \cos [2\pi f_c t + \theta(t)]
\]

\[
\hat{x}(t) = A(t) \sin [2\pi f_c t + \theta(t)]
\]

where \( A, \theta \) vary slowly compared with \( f_c \). Let

\[
z(t) := [x(t) + j\hat{x}(t)] e^{-j2\pi f_c t}
\]

then \( z(t) = A(t) e^{j\theta(t)} \) and \( |Z(f)| = 0, \quad |f| > \frac{1}{2} B_x \).
Proof

\[ x(t) + j\hat{x}(t) \leftrightarrow X(f) + j\hat{X}(f) = 2X(f)u(f) \]
\[ z(t) \leftrightarrow Z(f) = 2X(f + f_c)u(f + f_c) \]

Suppose \( z(t) = A(t)e^{j\theta(t)} \), then \( A, \theta \) vary slowly. Also

\[ x(t) = Re[z(t) \cdot e^{j2\pi f_c t}] = A(t)\cos(2\pi f_c t + \theta(t)) \]
\[ \hat{x}(t) = Im[z(t)e^{j2\pi f_c t}] = A(t)\sin(2\pi f_c t + \theta(t)) \]

Example

\[ f_c = 10, B_x = 2 \]

\[ z(t) = \mathcal{F}^{-1}[Z] = \int_{-\infty}^{\infty} Z(f)e^{j2\pi ft}df = 2\int_{-\infty}^{0} fe^{j2\pi ft}df \]
\[ = \ldots = \frac{1}{\pi f^2}\{e^{j2\pi f(1-2\pi it)} - 1\} \]
\[ = \frac{1}{\pi f^2}\{(\cos 2\pi ft + 2\pi t \sin 2\pi ft - 1) + j(\sin 2\pi ft - 2\pi t \cos 2\pi ft)\} \]

\[ A(t) = \frac{1}{\pi f^2}\{(\cos 2\pi ft + 2\pi t \sin 2\pi ft - 1)^2 + (\sin 2\pi ft - 2\pi t \cos 2\pi ft)^2\}^{1/2} \]

\[ \theta(t) = \tan^{-1}\frac{\sin 2\pi ft - 2\pi t \cos 2\pi ft}{\cos 2\pi ft + 2\pi t \sin 2\pi ft} \]
Demodulating NB-PM

\[
x(t) = \cos 2\pi f_c t - \phi_\Delta m(t) \sin 2\pi f_c t
\]

\[
y(t) = \cos 2\pi f_c t \cdot \sin 2\pi f_c t - \phi_\Delta m(t) [\sin 2\pi f_c t]^2
\]

\[
= -\frac{1}{2} \phi_\Delta m(t) + \frac{1}{2} \sin 4\pi f_c t + \frac{1}{2} \phi_\Delta m(t) \cos 4\pi f_c t
\]

where \(-\frac{1}{2} \phi_\Delta m(t) = z(t)\)

\[
Y(f) = -\frac{1}{2} \phi_\Delta M(f) + \frac{1}{4} \{ \delta(f - 2f_c) - \delta(f + 2f_c) \}
\]

\[
+ \frac{1}{4} \phi_\Delta \{ M(f - 2f_c) + M(f + 2f_c) \}
\]

“Wideband” PM/FM

Suppose \( m(t) = \begin{cases} 
A_m \sin 2\pi f_m t & \text{PM} \\
A_m \cos 2\pi f_m t & \text{FM}
\end{cases} \)

Then \( \Theta(t) = \begin{cases} 
A_m \phi_\Delta \sin 2\pi f_m t & \text{PM} \\
A_m f_m f_m^{-1} \sin 2\pi f_m t & \text{FM}
\end{cases} =: \beta \sin 2\pi f_m t, \quad \beta = \text{index of modulation}
\)

\[
x(t) = \cos [2\pi f_c t + \beta \sin 2\pi f_m t]
\]

\[
= \cos (2\pi f_c t) \cos (\beta \sin 2\pi f_m t) - \sin (2\pi f_c t) \sin (\beta \sin 2\pi f_m t)
\]

periodic with period \( f_m^{-1} \)
Exercise

\[ J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j[\beta \sin \lambda - n\lambda]} d\lambda \]

So for pure tone message signal:

\[ x(t) = J_0(\beta) \cos 2\pi f_c t \]
\[ + \sum_{n > 0, \text{even}} J_n(\beta) \{ \cos [2\pi (f_c + nf_m) t] + \cos [2\pi (f_c - nf_m) t] \} \]
\[ + \sum_{n > 0, \text{odd}} J_n(\beta) \{ \cos ([2\pi (f_c + nf_m) t] - \cos [2\pi (f_c - nf_m) t]) \} \]

**Approximate \( B_x \)**

For FM, define \( D = \frac{f_\Delta}{B_m} \) deviation ratio

For PM, define \( D = \phi_\Delta \)
Carson’s Rule: If \( D \ll 1 \) or \( D \gg 1 \), then
\[
B_x \cong 2(D + 1)B_m
\]

Example – For commercial FM broadcast

\[
B_m = 15 \text{ kHz, } f_\Delta = 75 \text{ kHz, } D = f_\Delta/B_m = 5
\]

So Carson’s rule gives
\[
B_x = 2 \times 6 \times 15 = 180 \text{ kHz}
\]

Actually \( B_x \approx 240 \text{ kHz} \)

Compare \( B_x \) for FM vs. AM!!

**FM-Demodulator**

\[
x(t) = \cos[2\pi f_c t + \theta(t)] \quad \text{LTI} \quad \rightarrow \quad H(f) \quad \rightarrow \quad y(t)
\]

Suppose \( H(f) = a + b2\pi|f| = a + b(j2\pi f)(-jsgn f) \)

\[
Y(f) = H(f)X(f) = aX(f) + b(j2\pi f)(-jsgn f)(X(f))
\]

\[
y(t) = ax(t) + b \frac{d}{dt}(\dot{\theta}(t)) = a \cos[2\pi f_c t + \theta(t)] + b \frac{d}{dt} \{ \sin(2\pi f_c t + \theta(t)) \}
\]

\[
= [a + b2\pi f_c + b\dot{\theta}(t)] \cos[2\pi f_c t + \theta(t)]
\]

where \( \dot{\theta}(t) = f_\Delta m(t) \)

\( a + b2\pi f_c + b f_\Delta m(t) \) obtained by envelope detection, then suppress dc to get \( bf_\Delta m(t) \):

\[
a + b2\pi f_c + bf_\Delta m(t)
\]