Chapter 13

Exercises

Each problem is annotated with the letter E, T, C which stands for exercise, requires some thought, requires some conceptualization. Problems labeled E are usually mechanical, those labeled T require a plan of attack, those labeled C usually have more than one defensible answer.

1. E Consider the signal \( x \) given by

\[ \forall n, \quad x(n) = \sin(\omega_0 n) u(n). \]

(a) Show that the Z transform is

\[ \forall z \in \text{RoC}(x), \quad \hat{X}(z) = \frac{z \sin(\omega_0)}{z^2 - 2z \cos(\omega_0) + 1}, \]

where

\[ \text{RoC}(x) = \{ z \in \text{Complex} \mid |z| > 1 \}. \]

(b) Where are the poles and zeros?

(c) Is \( x \) absolutely summable?

2. T Consider the signal \( x \) given by

\[ \forall n, \quad x(n) = a^{|n|}, \]

where \( a \in \text{Complex}. \)

(a) Find the Z transform of \( x \). Be sure to give the region of convergence.

(b) Where are the poles?

(c) Under what conditions is \( x \) absolutely summable?

3. E Consider a discrete-time LTI system with transfer function given by

\[ \forall z \in \{ z \mid |z| > 0.9 \}, \quad \hat{H}(z) = \frac{z}{z - 0.9}. \]
Suppose that the input $x$ is given by
\[ \forall n \in \text{Integers}, \quad x(n) = \delta(n) - 0.9\delta(n - 1). \]

Find the Z transform of the output $y$, including its region of convergence.

4. **E** Consider the exponentially modulated sinusoid $y$ given by
\[ \forall n \in \text{Integers}, \quad y(n) = a^n \cos(\omega_0 n)u(n), \]
where $a$ is a real number, $\omega_0$ is a real number, and $u$ is the unit step signal.

(a) Find the Z transform. Be sure to give the region of convergence. **Hint:** Use example ?? and section ??.

(b) Where are the poles?

(c) For what values of $a$ is this signal absolutely summable?

5. **T** Suppose $x \in \text{DiscSignals}$ satisfies
\[ \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty, \quad 0 < r_1 < r < r_2, \]
for some real numbers $r_1$ and $r_2$ such that $r_1 < r_2$. Show that
\[ \sum_{n=-\infty}^{\infty} |nx(n)r^{-n}| < \infty, \quad 0 < r_1 < r < r_2. \]

**Hint:** Use the fact that for any $\epsilon > 0$ there exists $N < \infty$ such that $n(1 + \epsilon)^{-n} < 1$ for all $n > N$.

6. **T** Consider a causal discrete-time LTI system where the input $x$ and output $y$ are related by the difference equation
\[ \forall n, \quad y(n) + b_1 y(n - 1) + b_2 y(n - 2) = a_0 x(n) + a_1 x(n - 1) + a_2 x(n - 2), \]
where $b_1, b_2, a_0, a_1, \text{ and } a_2$ are real-valued constants.

(a) Find the transfer function.

(b) Say as much as you can about the region of convergence.

(c) Under what conditions is the system stable?

7. **E** This exercise verifies the time delay property of the Laplace transform. Show that if $x$ is a continuous-time signal, $\tau$ is a real constant, and $y$ is given by
\[ \forall t, \quad y(t) = x(t - \tau), \]
then its Laplace transform is
\[ \forall s \in \text{RoC}(y), \quad \hat{Y}(s) = e^{-s\tau} \hat{X}(s), \]
with region of convergence
\[ \text{RoC}(y) = \text{RoC}(x). \]
8. **E** This exercise verifies the convolution property of the Laplace transform. Suppose $x$ and $h$ have Laplace transforms $\hat{X}$ and $\hat{H}$. Let $y$ be given by

$$\forall t, \quad y(t) = (x \ast h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau.$$ 

Then show that the Laplace transform is

$$\forall s \in \text{RoC}(y), \quad \hat{Y}(s) = \hat{X}(s)\hat{H}(s),$$

with

$$\text{RoC}(y) \supset \text{RoC}(x) \cap \text{RoC}(h).$$

9. **T** This exercise verifies the conjugation property of the Laplace transform, and then uses this property to demonstrate that for real-valued signals, poles and zeros come in complex-conjugate pairs.

(a) Let $x$ be a complex-valued continuous-time signal and $y$ be given by

$$\forall t, \quad y(t) = [x(t)]^*.$$ 

Show that

$$\forall s \in \text{RoC}(y), \quad \hat{Y}(s) = [\hat{X}(s^*)]^*,$$

where

$$\text{RoC}(y) = \text{RoC}(x).$$

(b) Use this property to show that if $x$ is real, then complex poles and zeros occur in complex conjugate pairs. That is, if there is a zero at $s = q$, then there must be a zero at $s = q^*$, and if there is a pole at $s = p$, then there must also be a pole at $s = p^*$.

10. **T** This exercise verifies the time scaling property of the Laplace transform. Let $y$ be defined by

$$\forall t \in \text{Reals}, \quad y(t) = x(ct),$$

for some real number $c$. Show that

$$\forall s \in \text{RoC}(y), \quad \hat{Y}(s) = \hat{X}(s/c)/|c|,$$

where

$$\text{RoC}(y) = \{s \mid s/c \in \text{RoC}(x)\}.$$ 

11. **E** This exercise verifies the exponential scaling property of the Laplace transform. Let $y$ be defined by

$$\forall t \in \text{Reals}, \quad y(t) = e^{at}x(t),$$

for some complex number $a$. Show that

$$\forall s \in \text{RoC}(y), \quad \hat{Y}(s) = \hat{X}(s - a),$$

where

$$\text{RoC}(y) = \{s \mid s - a \in \text{RoC}(x)\}.$$
12. Consider a discrete-time LTI system with impulse response
\[ h(n) = a^n \cos(\omega_0 n)u(n), \]
for some \( \omega_0 \in \text{Reals} \). Show that if the input is
\[ x(n) = e^{j\omega_0 n} u(n), \]
then the output \( y \) is unbounded.

13. Find and plot the inverse Z transform of
\[ \hat{X}(z) = \frac{1}{(z - 3)^3} \]
with
(a) \( \text{Roc}(x) = \{z \in \text{Complex} \mid |z| > 3\} \)
(b) \( \text{Roc}(x) = \{z \in \text{Complex} \mid |z| < 3\} \).

14. Obtain the partial fraction expansions of the following rational polynomials. First divide through if necessary to get a strictly proper rational polynomial.

(a) \( \frac{z + 2}{(z + 1)(z + 3)} \)

(b) \( \frac{(z + 2)^2}{(z + 1)(z + 3)} \)

(c) \( \frac{z + 2}{z^2 + 4} \)

15. Find the inverse Z transform \( x \) for each of the three possible regions of convergence associated with
\[ \hat{X}(z) = \frac{(z + 2)^2}{(z + 1)(z + 3)}. \]
For which region of convergence is \( x \) causal? For which is \( x \) strictly anti-causal? For which is \( x \) two-sided?

16. Find the inverse Z transform \( x \) for each of the two possible regions of convergence associated with
\[ \hat{X}(z) = \frac{z + 2}{z^2 + 4}. \]
17. E Consider a stable system with impulse response 
\[ h(n) = (0.5)^n x(n). \]
Find the steady-state response to a unit step input.

18. E Let \( h(n) = 2^n u(-n), \) all \( n, \) and \( g(n) = 0.5^n u(n), \) for all \( n. \) Find \( h \ast u \) and \( g \ast u, \) where \( u \) is the unit step.

19. This exercise shows how we can determine the transfer function and frequency response of an LTI system from its step response. Suppose a causal system with step input \( x = u, \) produces the output 
\[ \forall n \in \text{Integers}, \quad y(n) = (1 - 0.5^n)u(n). \]
(a) Find the transfer function (including its region of convergence).
(b) If the system is stable, find its frequency response.
(c) Find the impulse response of the system.

20. Consider an LTI system with impulse response \( h \) given by 
\[ \forall n \in \text{Integers}, \quad h(n) = 2^n u(n). \]
(a) Find the transfer function, including its region of convergence.
(b) Use the transfer function to find the Z transform of the step response.
(c) Find the inverse transform of the result of part (b) to obtain the step response in the time domain.