EE 121 – Introduction to Digital Communication Systems Spring 2016

Homework 2

Assigned: Tue, 2/9/16 Due: Tue, 2/16/16

Instructor: Omur Ozel *GSI: Preetum Nakkiran*

1 Sending a Bit in Continuous Time

We showed in the class that the detection problem in Q1 of HW1 is an abstraction of its continuous time counterpart. In this question, we explore this continuous time problem and how it reduces to the form in Q1 of HW1.

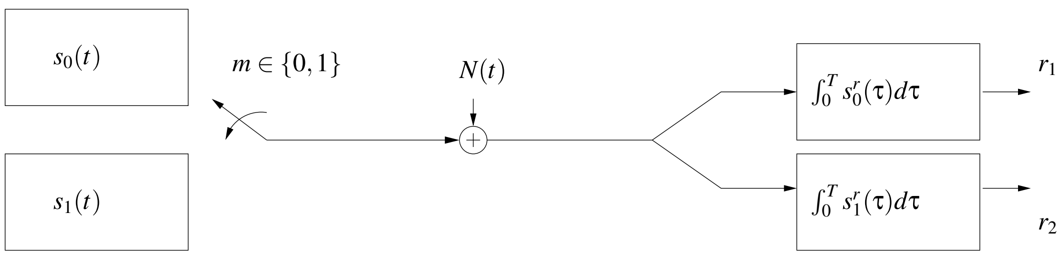

Figure 1: Setting for Question 1

Let us consider the setting in Figure 1. This is an additive noise channel where $N(t)$ is an additive Gaussian noise process with zero mean and power spectral density $S_N(f) = \frac{N_0}{2}$ W/Hz. The transmitter sends the message $m \in \{0,1\}$ with equal prior probability. The channel input is $s_0(t)$ if the message is $m = 0$ and $s_1(t)$ if the message is $m = 1$.The communication duration is T seconds. We specify $s_0(t)$ for $0 \le t \le T$ as follows:

$$s_0(t) := \begin{cases} \frac{1}{2}, & 0 \le t < \frac{T}{2} \\ -\frac{1}{2}, & \frac{T}{2} \le t \le T \end{cases}, \tag{1}$$

and $s_1(t) = -s_0(t)$.

(i) Let $s_0^r(t) = s_0(t)$ and $s_1^r(t) = s_1(t)$ for $0 \le t \le T$. Determine the optimal decoding rule given the vector $[r_1\ r_2]$. Calculate the probability of error for $N_0 = 1$ and $T = 1$.

(ii) Now, assume $s_1(t) = 1$ for $0 \le t \le T$. Determine the optimal decoding rule given the vector $[r_1\ r_2]$. Calculate the probability of error for $N_0 = 1$ and $T = 1$.

(iii) Now, assume the original $s_0(t), s_1(t)$ and $s_0^r(t) = s_0(t)$ whereas $s_1^r(t) = sin(2\pi t)$ for $0 \le t \le T$. Determine the optimal decoding rule given the vector $[r_1\ r_2]$. Calculate the probability of error for $N_0 = 1$ and $T = 1$.

(Hint: Use the fact that $n_0 = \int_0^T s_0^r(\tau)N(\tau)d\tau$ and $n_1 = \int_0^T s_1^r(\tau)N(\tau)d\tau$ are jointly Gaussian distributed random variables. Recall that it suffices to define jointly Gaussian random variables by their mean and covariance.)

2 Pulse Amplitude Modulation

Let the pulse shape $s(t)$ be

$$s(t) := \begin{cases} 1, & -\frac{T}{2} \le t < \frac{T}{2} \\ 0, & \text{o.w.} \end{cases}, \tag{2}$$

Consider the 4-level Pulse Amplitude Modulated signal

$$x_m(t) = A_m s(t - \frac{T}{2}) \tag{3}$$

where $A_m = -3$ if $m = 1$, $A_m = -1$ if $m = 2$, $A_m = 1$ if $m = 3$ and $A_m = 3$ if $m = 4$. We assume $m \in \{1, 2, 3, 4\}$ with equal probability.

$x_m(t)$ passes through an additive white Gaussian noise channel where the noise is denoted as $N(t)$ and its power spectral density is $\frac{N_0}{2}$. The receiver observes $Y(t) = x_m(t) + N(t)$

(a) Consider the related random process $V(t) = \sum_{n=-\infty}^{\infty} \tilde{A}_n s(t - nT - \frac{T}{2})$ where $\tilde{A}_n$ is an independent and identically distributed discrete time random process that takes value in $\{-3, -1, 1, 3\}$ with equal probability. Determine the autocorrelation function $R_V(t + \tau; t) = \mathbb{E}[V(t + \tau)V(t)]$ and the average autocorrelation function $\overline{R}_V(\tau) = \frac{1}{T} \int_0^T R_V(t + \tau; t)dt$. Determine the power spectral density of this process as the Fourier transform of $\overline{R}_V(\tau)$.

(b) Assume that the received signal $Y(t)$ is filtered as $r = \int_0^T Y(\tau)s(\tau)d\tau$ and the receiver performs decoding based on r. Determine the optimal decoding rule and corresponding probability of error in terms of the Q-function.