Discrete Fourier Series

C.T.F.T. \[ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

D.T.F.T. \[ X(n) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi n} \]

D.F. Series. \[ X(k) = \sum_{n} x(n) e^{-j2\pi nk/N} \]

D.F.T. \[ X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \]
**DFS - Discrete Fourier Series**

Deal with $x(n)$ periodic, discrete time signal.

$$x(n) = x(n + kN)$$

any integer = period.

Idea: Decompose $x(n)$ in terms of exponentials.

periodic with period $N$.

$$e_{k}(n) = e^{j2\pi nk/N} \quad \forall \ n \in \mathbb{Z}$$

There are $N$ periodic exponentials with period $N$.

$$\hat{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{-j2\pi nk/N}$$

weights
\( e_{\pi}(x) \) is periodic with period \( \pi \) if \( \pi + \pi = \pi \) or \( e^x + e^x = e^x \) arbitrary.

Proof:

\[
2 \frac{2\pi}{e} = e^{2\pi} \Rightarrow e^{2\pi} = e^{2\pi} = e^{2\pi} = e^{2\pi}
\]
How find "weight" \( X(k) \)?

**Proposal:**

\[
X(k) = \sum_{n=0}^{N-1} X(n) e^{-j2\pi nk/N}
\]

**Proof:**

\[
X(k) = \sum_{n=0}^{N-1} \left( \frac{1}{N} \sum_{l=-N/2}^{N/2} X(l) e^{j2\pi l n/N} \right) e^{-j2\pi kn/N}
\]

\[
= X(0) + \sum_{l=1}^{N-1} X(l) e^{-j2\pi l k/N}
\]

What is \( A \)?

\[
\sqrt{2} \sum_{l=0}^{N-1} X(l) S(l-k-rN) = X(k+rN)
\]
Case (i): If \( k \) is an int. multiple of \( N \).

\[ l - k = rN \]

\[ \sum_{n=0}^{\infty} \frac{e^{j2\pi rNn}}{N} = 1 \]

Case (ii): \( l - k \neq rN \) and \( k \) is not an int. multiple of \( N \).

\[ A = \sum_{n=0}^{\infty} e^{j2\pi (l-k)n} \]

Recall \( \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \)

\[ A = \frac{1}{N} \]

\[ A = \delta(l-k-rN) \]
\[ X(k) = X(k+rN) \]

\[ Y = \text{arb. int.} \]

\[ \Rightarrow \]

\[ \text{From now on refer to } X(k) \]
DFS
\[ X(k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk} \]

Analysis:
DFS

\[ X(n) = \sum_{k=-\infty}^{\infty} x(k) e^{j2\pi nk} \]

Synthesis:
DFS

Periodic point

Perodic

\[ X \]
Shift Property

\[
\hat{x}(n) \xrightarrow{\text{d}} \frac{\hat{x}(k)}{e^{-j2\pi nk/N}} \xrightarrow{\text{d}} \hat{x}_3(k)
\]

Periodic Convolution:

\[
\hat{x}_1 \ast \hat{x}_2 = \hat{x}_3 \quad \text{period } N
\]

\[
\hat{x}_3(k) = \hat{x}_1(k) \hat{x}_2(k)
\]
The Discrete Fourier Transform (DFT) is given by:

\[ X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \]

where \( x(n) \) is the input sequence, \( X(k) \) is the output sequence, and \( N \) is the length of the sequence. The DFT maps a sequence of \( N \) complex numbers to another sequence of \( N \) complex numbers.
First Approach To DFT via DFS

1. Start with a finite discrete seq \( x(n) \)

   \( N \) points long \( n = 0, \ldots, N-1 \)

2. "Periodicize" \( x(n) \) to get \( \hat{x}(n) \) with

   \( R_N(n) \)

   \[ R_N(n) = \begin{cases} 1 & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases} \]

   \[ \hat{x}(n) = \sum_{k=-\infty}^{+\infty} x(n + kN) \]
3. Take DFS of \( x(n) \) → \( \hat{X}(k) \)

4. Take one period of \( \hat{X}(k) \) to get

\[
\hat{X}(k) = \text{DFT of } x(n)
\]

\[
\hat{X}(k) = X(k) R_{Npt}(k)
\]

\[\begin{align*}
X(n) &\xrightarrow{Npt} X(n) \quad \text{periodic} \\
\text{DFS} &\xrightarrow{Npt} \hat{X}(k) \quad \text{Periodic Npt} \\
\end{align*}\]
\[ X(k) = N \cdot \text{DFT} \{ x(n) \} = \sum_{n=0}^{N-1} x(n) \cdot e^{-j \frac{2\pi nk}{N}} \quad 0 \leq k < N \]

\[ x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j \frac{2\pi nk}{N}} & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \]
\[ X_k = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} & 0 \leq k < N \\ 0 & \text{otherwise} \end{cases} \]

DFT is equally spaced samples of DTFT.
Given \( \Omega = \frac{2\pi}{K} \) for \( K = 0, 1, 2, 3, 4, 5, 6, 7 \), the \( 8 \) pt. DFT. A plot is shown for a 2D plane with a circular axis.
\[ X(n) \overset{\text{int}}{\rightarrow} \hat{X}(n) \overset{\text{finite extent}}{\rightarrow} X(k) \]