Nov 16, 05

Discrete Fourier Series

$x(t)$ ← real. continuous $\xrightarrow{\text{C.T.F.T.}}$ $X(\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$ (Ω ← real.) C.T.F.T

$x(n)$ ← int. discrete time signal (or finite length) $\xrightarrow{\text{D.T.F.T}}$ $X(\omega) = \sum_n x(n)\, e^{-j\omega n}$. ($\omega$ ← real.)

$x(n)$ ← int, discrete time (or finite length) $\xrightarrow{\text{Z.T}}$ $X(z) = \sum_n x(n)\, z^{-n}$ (z ← complex.)

$\tilde{x}(n)$ ← int, periodic. $\xrightarrow{\text{D.F. Series.}}$ $\tilde{X}(k) = \sum_{n \text{ over one period}} \tilde{x}(n)\, e^{-j\frac{2\pi nk}{N}}$

$x(n)$ ← int, finite length. $\xrightarrow{\text{D.F.T.}}$ $\underline{X(k)}$ ← int, finite length $= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}$

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DFS = Discrete Fourier Series

Deals with $\tilde{x}(n)$ periodic, discrete time signal.

$$\tilde{x}(n) = \tilde{x}(n + kN)$$

any integer = period.

Idea: Decompose $\tilde{x}(n)$ in terms of exponentials periodic with period N.

$$e_k(n) = e^{j\frac{2\pi nk}{N}} \qquad k = 0, \ldots, N-1$$

$Re[e_k(n)]$

There are N, periodic exponentials with period N.

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\frac{2\pi nk}{N}}$$

weights

$e_k(n)$ is periodic with period N:

$$e_k(n) \overset{??}{=} e_{k+rN}(n)$$

arbitrary int.

proof:

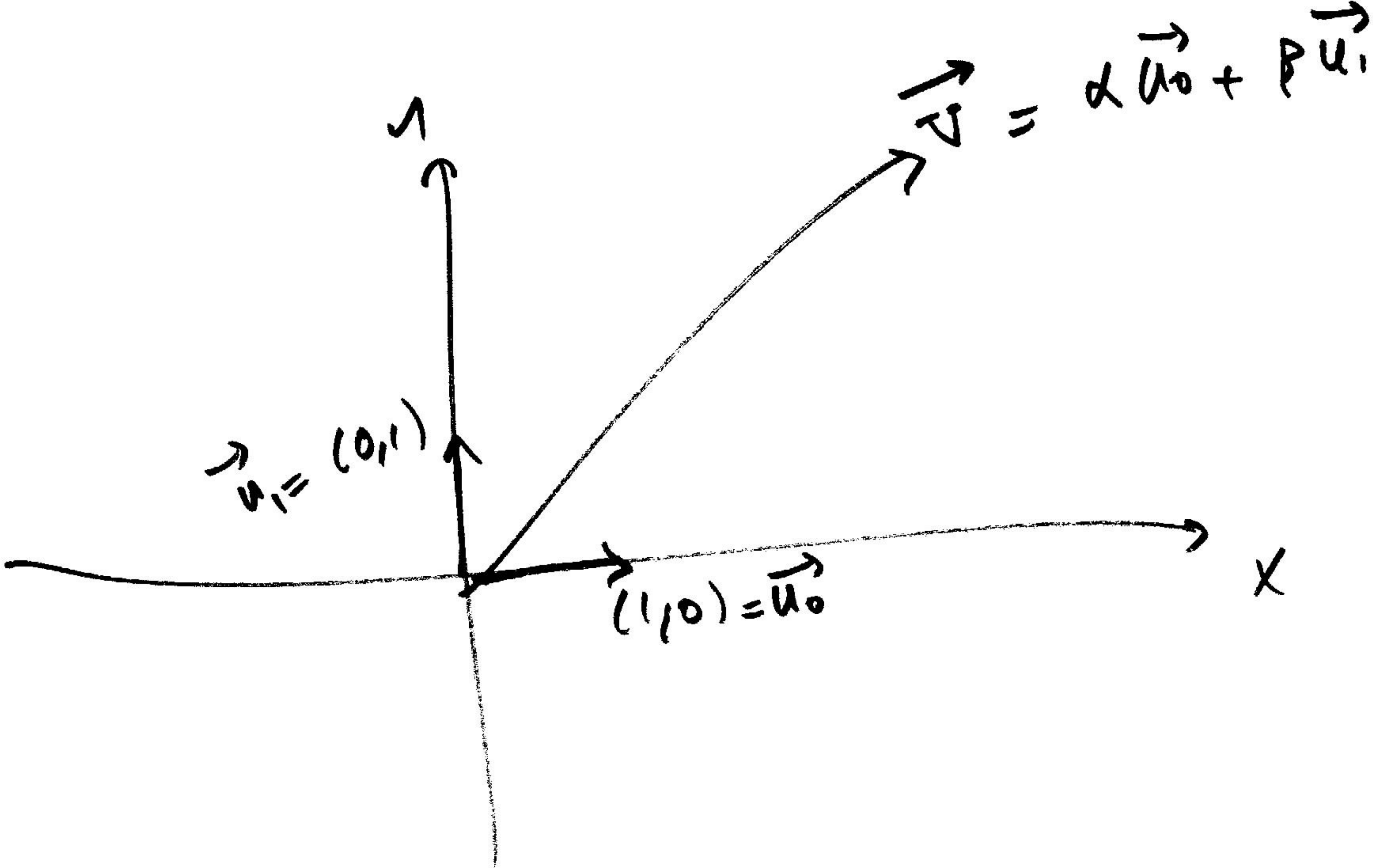
$$e^{j2\pi \frac{nk}{N}} \overset{?}{=} e^{j2\pi n\left(\frac{k+rN}{N}\right)}$$

$$= e^{j2\pi\frac{kn}{N}} e^{j2\pi n \frac{rN}{N}}$$

$$= e^{j2\pi\frac{kn}{N}}$$

$$e_0(n) = e_N(n) = e_{2N}(n) = e_{3N}(n) = \ldots$$

$$e_1(n) = e_{N+1}(n) = e_{2N+1}(n) = \ldots$$



$\vec{v} = \alpha \vec{u_0} + \beta \vec{u_1}$
y
$\vec{u_1} = (0,1)$
$(1,0) = \vec{u_0}$
x

Q How find "weights"? $X(k)$?

proposal:

$$X(k) \stackrel{??}{=} \sum_{n=0}^{N-1} \underbrace{\hat{x}(n)}\, e^{-j\frac{2\pi n k}{N}}$$

proof:

$$X(k) \stackrel{??}{=} \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{\ell=0}^{N-1} X(\ell)\, e^{j\frac{2\pi n \ell}{N}} \right) e^{-j\frac{2\pi n k}{N}}$$

$$\stackrel{??}{=} \sum_{\ell=0}^{N-1} X(\ell) \underbrace{\left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi(\ell-k)n}{N}} \right]}_{A}$$

$$A = \delta(\ell - k - rN)$$

what is A?

$$\checkmark \stackrel{??}{=} \sum_{\ell=0}^{N-1} X(\ell)\,\delta(\ell-k-rN) = X(k + rN) = X(k)$$

$r \to$ arb. int.

$X(k+rN) = X(k)$

Case (1): If $l-k$ is an int. multiple of N.

$$l-k = rN$$

$$(A) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi r N n}{N}} = 1$$

Case (2) $l-k \neq rN$ $\quad$ $l-k$ is not an int. multiple of N.

$$(A) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi\frac{(l-k)n}{N}}$$

Recall $\displaystyle\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$$(A) = \frac{1}{N}\,\frac{1-e^{j2\pi\frac{(l-k)N}{N}}}{1-e^{j2\pi\frac{(l-k)}{N}}} = \phi$$

$$(A) = \delta(l-k-rN)$$

$\bar{X}(K) = X(K + rN)$ $\quad r =$ arb. int.

$\Rightarrow$ $X(K)$ is a periodic sequence with period N.

$\Rightarrow$ From now on refer to $X(K)$ as

$\breve{X}(K)$

DFS pair.

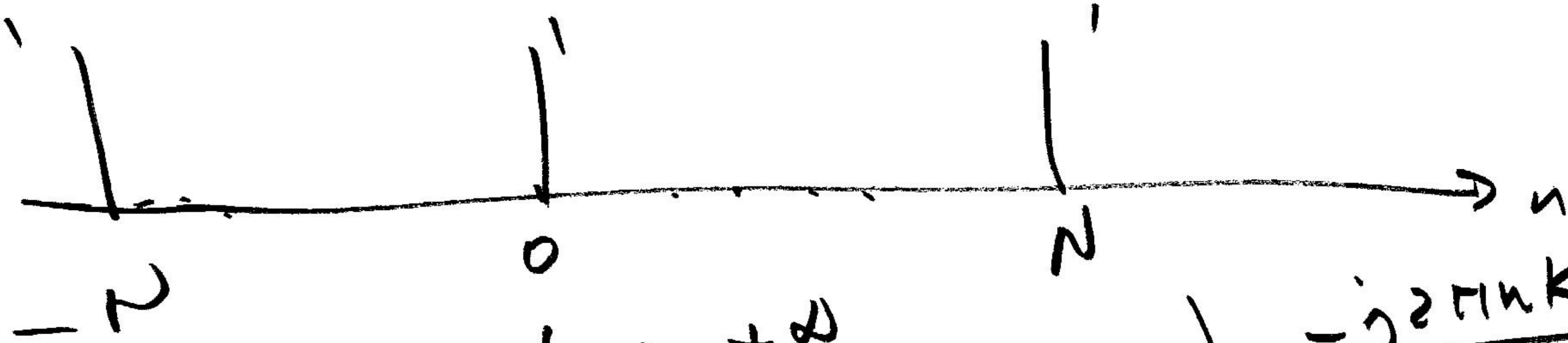
Analysis. $$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi n k}{N}}$$

synthesis $$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{+j\frac{2\pi n k}{N}}$$

DFS:

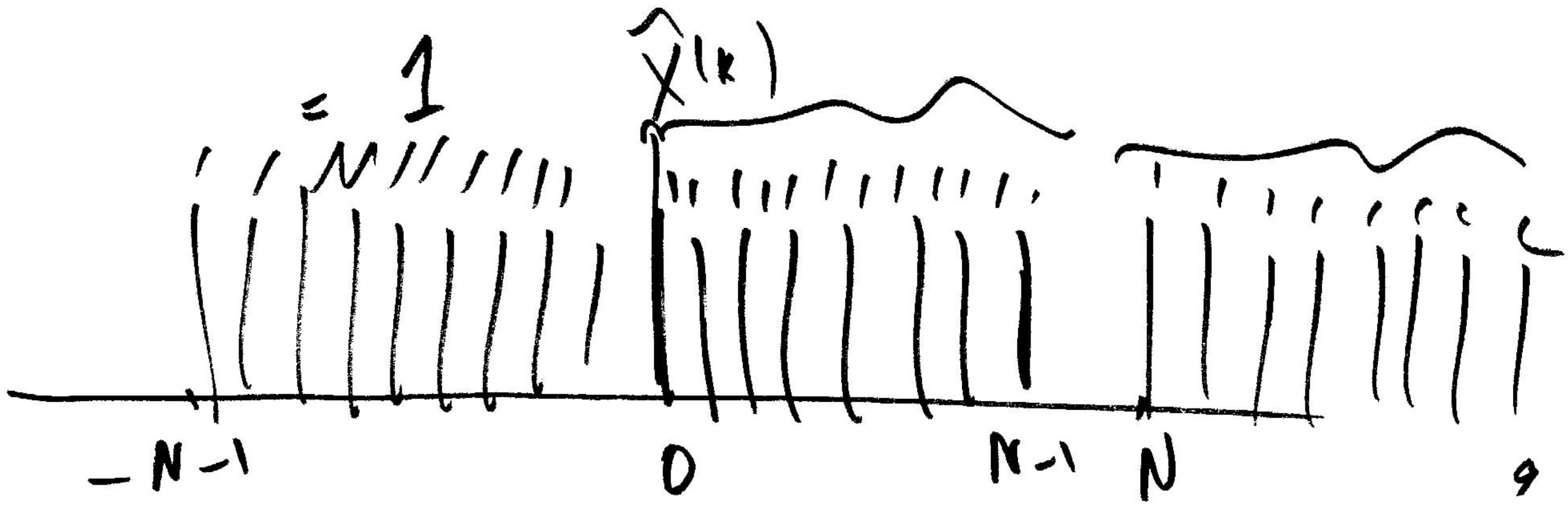
periodic N pt seq $\tilde{x}(n)$ $\longrightarrow$ periodic N point seq in freq. Domain. $\tilde{X}(k)$

Ex $\tilde{x}(n) = \sum_{r=-\infty}^{\infty} \delta(n + rN)$



$$\tilde{X}(k) = \sum_{n=0}^{N-1} \left(\sum_{r=-\infty}^{+\infty} \delta(n+rN) \right) e^{-j\frac{2\pi nk}{N}}$$

$= 1$



$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi nk}{N}}$$

equation train.

$$\tilde{x}(n) = \sum_{r=-\infty}^{+\infty} \delta(n + rN)$$

Shift Property

$$\tilde{x}(n) \longleftrightarrow \tilde{X}(k)$$

$$\tilde{x}(n-n_0) \longleftrightarrow e^{-j\frac{2\pi n_0 k}{N}} \tilde{X}(k)$$

Periodic Convolution:

$$\underset{\text{period } N}{\tilde{x}_1} \underset{\text{period convol}}{\circledast} \underset{\text{period } N.}{\tilde{x}_2} = \tilde{x}_3 \rightarrow \text{period } N.$$

$$\triangleq \sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m)$$

$$\tilde{X}_3(k) = \tilde{X}_1(k)\, \tilde{X}_2(k)$$

$\underline{DFT}$ = Discrete Fourier Trafo.

Npt

$x(n) \longrightarrow X(k)$ — Npt seq

DFS

DFT

DTFT.

First Approach To DFT via DFS

1. Start with a finite extent seq $x(n)$ N points long $n=0,\dots,N-1$

2. "Periodicize" $x(n)$ to get $\tilde{x}(n)$ with period N.

$$x(n) = \tilde{x}(n) R_N(n)$$

extracts one period of $\tilde{x}(n)$

$$R_N(n) = \begin{cases} 1 & n=0,\dots,N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{x}(n) = \sum_{r=-\infty}^{+\infty} x(n+rN)$$

periodicization

3. Take DFS of $\tilde{x}(n) \rightarrow \tilde{X}(k)$

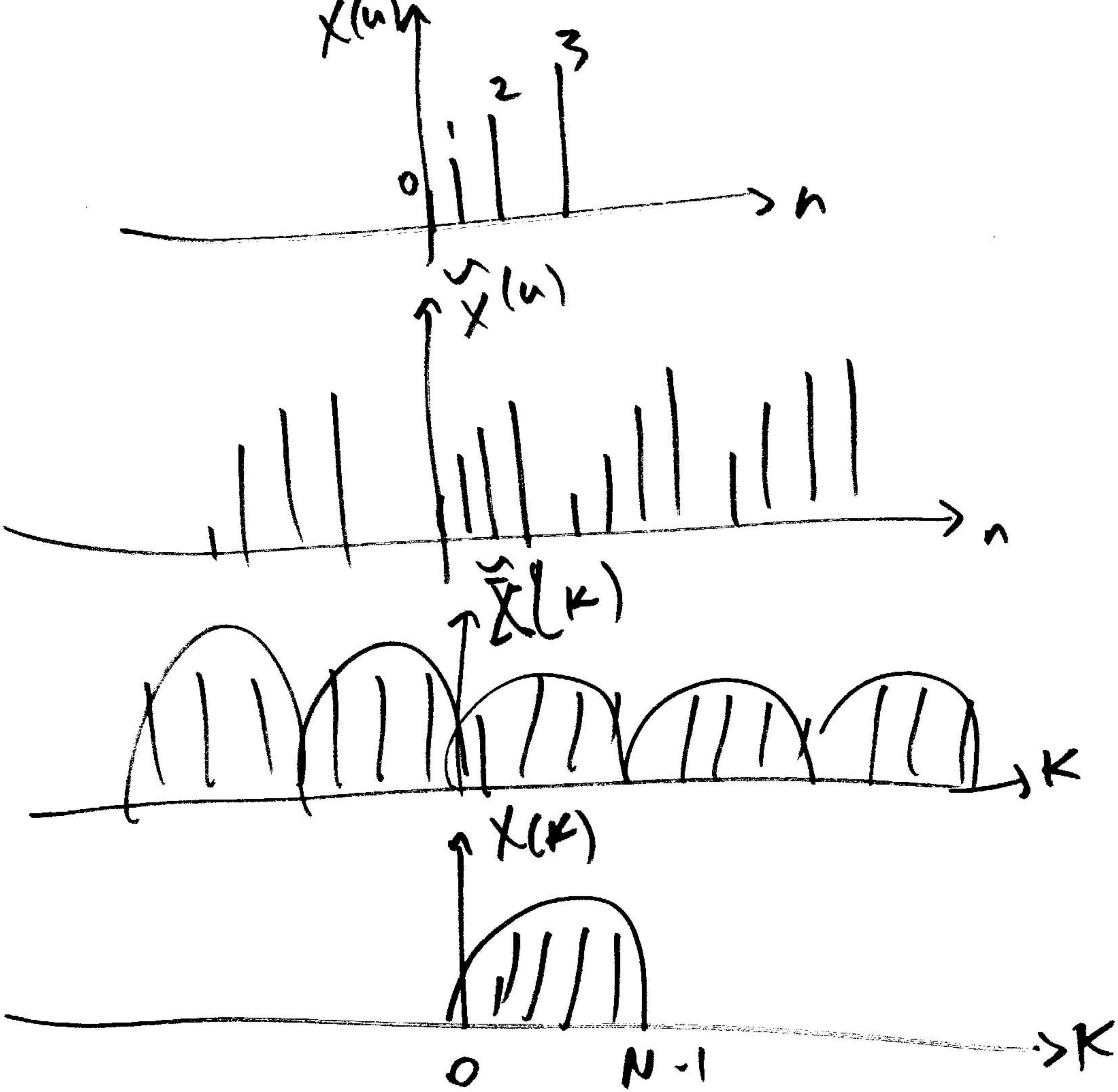
4. Take one period of $\tilde{X}(k)$ to get

$$X(k) = \text{DFT of } x(n)$$

$$X(k) = \tilde{X}(k)\, R_N(k)$$

$$\underset{N\text{pt}}{x(n)} \longrightarrow \underset{\substack{\text{periodic} \\ N\text{pt.}}}{\tilde{x}(n)} \xrightarrow{\text{DFS}} \underset{\substack{\text{periodic} \\ N\text{pt}}}{\tilde{X}(k)} \longrightarrow \underset{N\text{pt.}}{X(k)}$$

Ex



Defn of DFT

$$X(k) = N\text{pt DFT of } x(n) = \begin{cases} \displaystyle\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}} & 0 \le k < N \\ 0 & \text{otherwise} \end{cases}$$

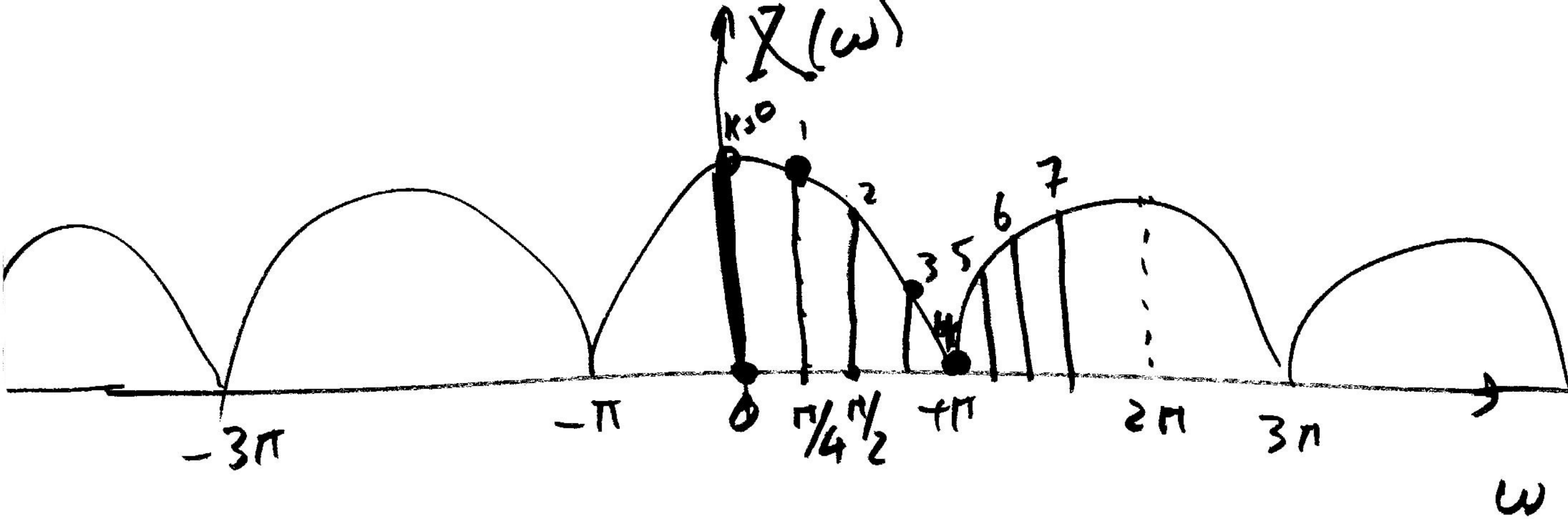
$$x(n) = \begin{cases} \displaystyle\frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi nk}{N}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Relate DFT To DTFT:

$$X(\omega) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$$X(k) = \begin{cases} \left[X(\omega)\right]_{\omega = \frac{2\pi k}{N}} & 0 \le k < N \\ 0 & \text{Otherwise.} \end{cases}$$

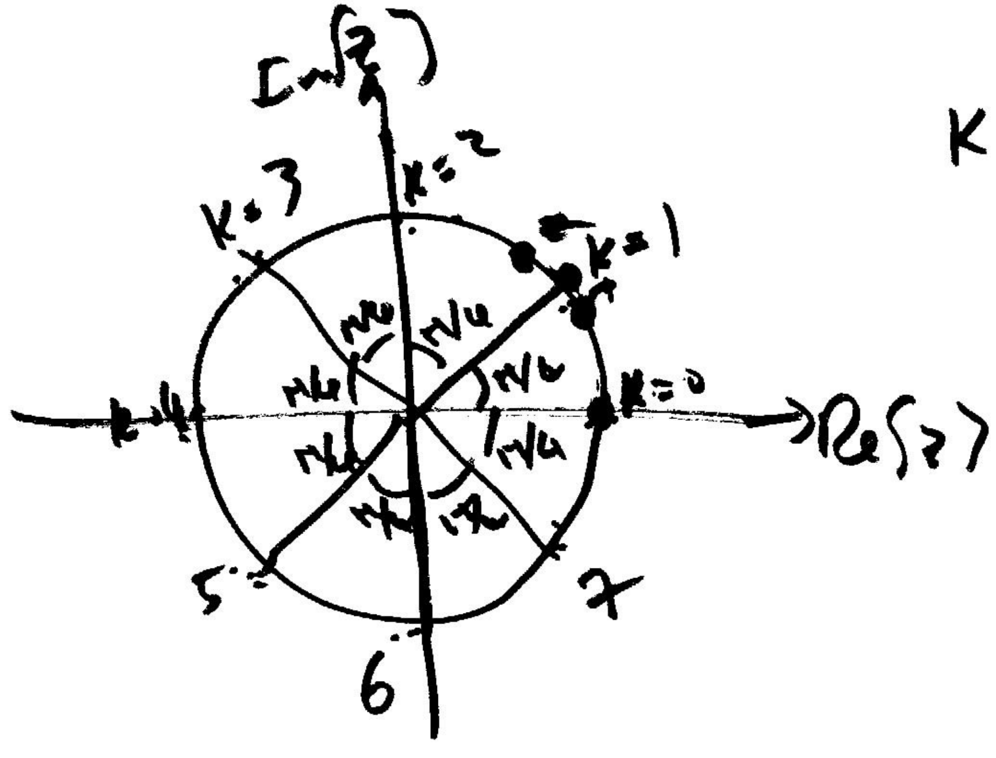
DFT is equally spaced samples of DTFT.



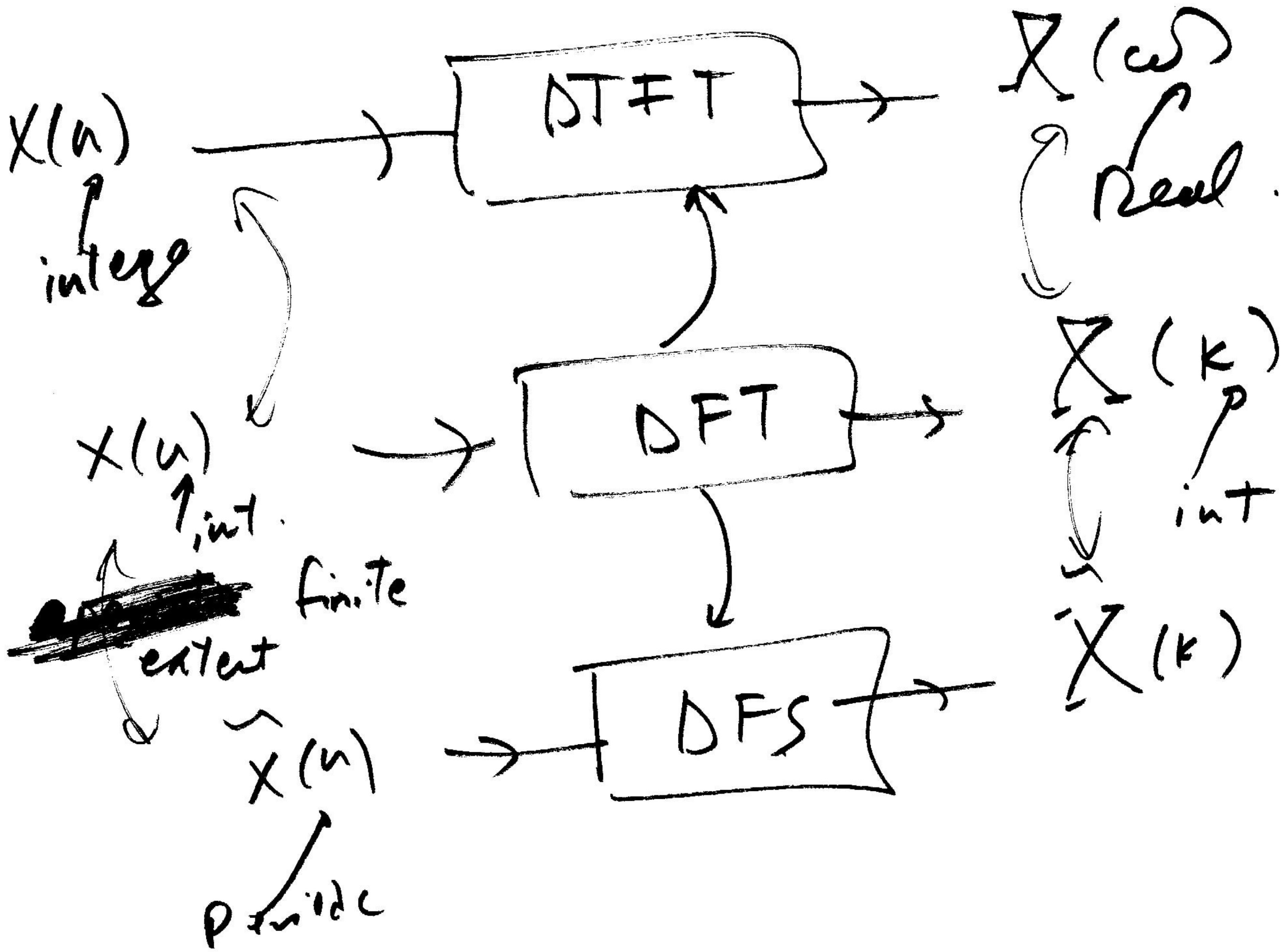
8 pt seq. → 4 pt DFT.

$$\omega = \frac{2\pi K}{8}$$

$K = 0, 1, 2, 3, 4, 5, 6, 7$



z plane


x(n)
integer
DTFT
X(ω)
Real
x(n)
int.
finite
extent
DFT
X(k)
int
x̃(n)
periodic
DFS
X̃(k)