Cascade Parallel Implementation of IIR filters with Rational Transfer Function.

Factoid: If coefficients of polynomial in one variable are real, then roots are either real or they are complex conjugate.

⇒ If $z_0$ is root, then so is $\bar{z}_0$.

- polyn of deg 2: $a z^2 + b z + c$
  - B.T. real
  - a pair of complex conjugate
- Poly of deg 3

\[ \alpha z^3 + \beta z^2 + \delta z + \delta \]

3 real roots

\[ k(z - a)(z - b)(z - c) \]

One real root, a pair of complex conjugate roots.

Cannot have 2 real roots and one complex.
A polynomial of degree 4: \( ax^4 + bx^3 + cx^2 + dx + e \)

- 4 real roots
- 2 real roots
- A pair of complex conjugates

Factors:
\[ k(2-20)(2-21)(2-22)(2-23) \]

**Conclusion:** Polynomial with real coefficients of odd degree always has a real root.

**Claim:** For a polynomial with real coefficients, I can factor it this way.
\[ P(z) = \prod_k (1 - c_k \bar{z}^i) \prod_k (1 - d_k \bar{z}^i) \prod_k (1 - f_k^* \bar{z}^i) \prod_k (1 - f_k \bar{z}^i) \]

\[ H(z) = \mathbb{E} \sum_{k=0}^{\infty} b_k z^{-k} \]

\[ H(z) = A \frac{\prod_k (1-e_k \bar{z}^i) \prod_k (1-f_k \bar{z}^i) (1-f_k^* \bar{z}^i)}{\prod_k (1-c_k \bar{z}^i) \prod_k (1-d_k \bar{z}^i) (1-d_k^* \bar{z}^i)} \]
Note: \[(1 - d_k \bar{z}^k)(1 - d_k^* \bar{z}^k)\]

\[= 1 - 2 \text{Re} \left[ d_k \right] \bar{z}^k + |d_k|^2 \bar{z}^{2k} \]

polynomial with real coeff.

\[= 1 + \beta_{1k} \bar{z}^k + \beta_{2k} \bar{z}^{2k} \]

Generic 2nd order:

\[H(z) = \frac{\prod_k H_k(z)}{H_0(z)}\]

\[H_k(z) = \frac{1 + \beta_{1k} \bar{z}^k + \beta_{2k} \bar{z}^{2k}}{1 - \alpha_{1k} \bar{z}^k - \alpha_{2k} \bar{z}^{2k}} = \frac{Y(z)}{X(z)}\]

\[\beta_{1k}, \beta_{2k}, \alpha_{1k}, \alpha_{2k} \text{ are all real.}\]
Q. How to implement $H_k(z)$.

$$H_k(z) = \frac{1 + \beta_{ik} z^{-1} + \beta_{2k} z^{-2}}{1 - \alpha_{ik} z^{-1} - \alpha_{2k} z^{-2}}$$

Cascade implementation

Direct Form II implementation of $H_k(z)$
\[ H(z) = \frac{1 - z^{-1}}{1 + \frac{1}{2} z^{-1} (1 + \frac{1}{2} z^{-1} + \frac{1}{2} z^{-2})} \]

\[ H(z) = H_1(z) \cdot H_2(z) \]

where

\[ H_1(z) = \frac{1 - z^{-1}}{1 + \frac{1}{2} z^{-1}} \]

\[ H_2(z) = \frac{1}{1 + z^{-1} + \frac{1}{2} z^{-2}} \]

\[ H_1(z) \]

\[ H_2(z) \]
$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 - \sum_{k=1}^{p} a_k z^{-k}} = \sum_k H_k(z)$$

$$H(z) = \sum_k A_k z^{-k} + \sum_k \frac{B_k}{1 - g_k z^{-1}} + \sum_k \frac{C_k + D_k z^{-1}}{1 - h_k z^{-1}}$$
$H(z) = 2 z^{-1} + \frac{1 - z^{-1}}{1 + z^{-1} + \frac{1}{2} z^{-2}}$
$H(z) = \sum_k H_k(z) + \prod_k G_k(z)$
Polynomial Interpolation

Fundamental Theory of Algebra

\[ P(z) = \beta_0 z^n + \beta_{n-1} z^{n-1} + \cdots + \beta_1 z + \beta_0 \]

Any random samples/values of an \( n \)th degree polynomial can be used to uniquely reconstruct it.
2D Polynomial Interpolation

Does not work

\[ p(z_1, z_2) = \alpha z_1^2 + \beta z_2^2 + \delta z_1 z_2 + \epsilon z_1 + \eta z_2 + \delta \]
Transposition Theorem.

Change order of input/output; change the direction of flow graph $\rightarrow$ get same system i.e.
same input/output relationship.

\[
H_k(z) = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 - \alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}}
\]
\[
\begin{cases}
\begin{align*}
r(n) &= d_{1k} y(n) + \beta_{1k} x(n) + \beta_{2k} x(n-1) \\
&+ a_{2k} y(n-1) \\
y(n) &= x(n) + r(n-1)
\end{align*}
\end{cases}
\]
\[
Y(n) = x(n) + d_{1k} y(n-1) + \\
\beta_{1k} x(n-1) + \beta_{2k} x(n-2) + d_{2k} y(n-2)
\]

\[
\frac{Y(z)}{X(z)} = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 - d_{1k} z^{-1} - d_{2k} z^{-2}}
\]

\[
= H(z)
\]