FIR Filter Design using Windows

Kaiser Window:

\[ W(n) = \begin{cases} I_0 \left( \beta \left( 1 - \left( \frac{n - \alpha}{\alpha} \right)^2 \right)^{\frac{1}{2}} \right) & \text{for } 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases} \]

\( I_0 = \text{zeroth order modified Bessel function.} \)

\( I_0(x) = 1 + \frac{x^2}{2^2 (1!)} + \frac{x^4}{2^4 (2!)^2} + \frac{x^6}{2^6 (3!)^2} + \ldots \)

Solve to the following diff. equ.:

\[ \frac{x^2}{dx^2} \frac{dy}{dx} + x \frac{dy}{dx} - \left( x^2 + n^2 \right) y = 0 \]

\( n^{\text{th}} \text{ order modified Bessel fn.} \)
$B'$ controls the shape of Kaiser window allowing trade off between sidelobe and mainlobe.

**Design Using Kaiser Window**

1. $\Delta W = W_s - W_p$: Transition width.
2. ripple = 8 $\rightarrow$ $A = -20 \log_{10} 8$

Choose $\alpha = \frac{M - 1}{2}$ and $B'$ as follows:

$M = 2\alpha = \frac{A - 8}{2.285 \Delta W}$
\[ \beta = \begin{cases} 
0.1102(A - 8.7) & A > 50 \\
0.5842(A - 21)^{0.4} & 21 \leq A < 50 \\
0 & A < 21 
\end{cases} \]
Linear phase filter.

\[ H_d(w) \rightarrow h_d(n) \]

\[ H(w) = H_m(w) \]

\[ \beta = 0 \rightarrow \text{Type I or Type II} \rightarrow \text{Both capable of LPF.} \]

Specification:

\[ W_p = 0.4 \pi \]

\[ W_s = 0.6 \pi \]
\[ \Delta w = \omega_s - \omega_p = 0.2 \pi \]
\[ A = -20 \log_{10} \delta = -60 \]
plug in (a), (b): To compute \( M, P' \)
\[ M = 3.7 \quad \Rightarrow \quad \# \text{up Taps} = M + 1 = 38 \Rightarrow \text{Type II} \]

Figure 7.25

Windows: No fine grain control over \( \delta_s, \delta_p, \Delta w \).
OPTIMAL FIR FILTER DESIGN

TYPE I  Generalized Linear Phase Filter

\[ H(\omega) = H_m(\omega) e^{j\beta} e^{-j\omega} \]

LPF

Assume Type I  \( \rightarrow \beta = 0 \),

\[ h(n) = h(N-n-1) \]

\[ H_m(\omega) = \left| \sum_{n=0}^{M} a(n) \cos (\omega n) \right| = G(\omega) \]

\[ H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \]

\[ H(\omega) = G(\omega) e^{-j\omega} \]

\[ N = \# \text{ taps is odd,} \]

\[ = 2M + 1 \]
Observations on $g(w)$

1. $g(w)$ is a continuous function and is as many times differentiable as you want.

2. How many local extrema does $g(w)$ have?

Extrema:
- Local maxima
- Local minima
- Global maxima
Express \( \cos(wn) \) as sum of powers of \( \cos(w) \):

\[
\cos(2w) = 2 \cos^2 w - 1
\]

\[
\cos(3w) = \cos(2w + w) = \cos 2w \cos w - \sin 2w \sin w
\]

\[
= \cos w [2 \cos^2 w - 1]
\]

\[
-2 \sin^2 w \cos w
\]

\[
= 2 \cos^3 w - \cos w
\]

\[
-2 \cos w [1 - \cos^2 w]
\]

\[
= 4 \cos^3 w - 3 \cos w
\]

Generally, \( \cos(wn) \) as sum of powers of \( \cos(w) \):

\[
\cos(wn) = \sum_{i=0}^{n} c_i \cos^n w
\]

\( \text{Chebyshev} \)

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\[ G(\omega) = \sum_{n=0}^{M} a(n) \left[ \sum_{i=0}^{\eta} \eta_i (\cos \omega)^i \right] \]

\[ G(\omega) = \sum_{n=0}^{M} \delta(n) (\cos \omega)^n \]

\[ \delta \text{ depends on } \eta \text{ and } \alpha \]

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To compute a local extremum of \( G(\omega) \), take the derivative and set to zero.

\[ \frac{dG(\omega)}{d\omega} = 0 \Rightarrow \sum_{n=0}^{M} \delta(n) n (\cos \omega)^{n-1} (-\sin \omega) \]

\[ = 0 \quad \Rightarrow \]
\[ \sin w = 0 \implies w = 0, \pi \]

\[ \sum_{n=0}^{M} \delta(n) n \left( \cos w \right)^{n-1} = 0 \implies \text{Max of } \frac{M-1}{M-1} \text{ Zeos.} \]

\text{Polynomial in } \cos w = x

\[
\Rightarrow \quad \text{total number of local extrema for } b(w) \text{ is }
(M-1) + 2 = M + 1
\]
Problem A

\[ |H(\omega)| \]

Given \( w_s, w_p, \delta_1, \delta_2 \), determine \( \omega(\omega) \) i.e. \( a(\omega) \) such that \( M \) is minimized.
Problem B: Given \( W_p, W_s, M, K = \frac{S_1}{\delta_2} \), find \( a(n) \) such that \( \delta_2 \) is minimized.

Show that if I had a box \( [A + B] \) that solves Problem B, then I can use it to solve Problem A.
Given $w_0, w_1, \delta_1, \delta_2$

Compute $k = \frac{\delta_1}{\delta_2}$, Guess $M$

ALG $B$

If $\delta_2 < \delta_2$

Is $\delta_2 < \delta_2$?

Yes

No

Increase $M$ by 1

Stop

Yes

Reduce $M$ by 1

Stop
Problem C: \[ E(\omega) = W(\omega) \left( G(\omega) - D(\omega) \right) \]

where \( W(\omega) \): positive weighting function \[ \frac{1}{k} \]

\[ G(\omega) = \sum_{n=0}^{N} a(n) e^{jn\omega} \]

\( D(\omega) \): Desired Freq. Response \[ \begin{cases} 1 & I_1 \\ 0 & I_2 \end{cases} \]

Problem C: Find \( a(n) \) To minimize \[ \max_{\omega \in F} |E(\omega)| \]

when \( F = I_1 \cup I_2 \)

is a subset of a closed interval \( 0 \leq \omega \leq \pi \)

\( I_1 = \text{Passband}, \quad I_2 = \text{Stopband} \)