Problem B: Given $w_s, w_p, K = \frac{S_1}{S_2}$, $M$

find $a(n)$ to minimize $S_2$

$$G(w) = \sum_{n=0}^{M} a(n) \cos(\omega n)$$

**Alt. Thm.**

$$E(w_i) = \pm S_2 \quad i = 1, \ldots, M+2$$

$$E(w_i) = -E(w_{i+1}) \quad i = 1, \ldots, M+1$$

$$W(w_i) [G(w_i) - D(w_i)] = (-1)^i S_2.$$
Show if we know \( w_i \), how to find \( a(n) \) and \( \delta_2 \)?

\[
G(Wi) = \sum_{n=0}^{M} a(n) \cos(w_i n)
\]

\[
G(Wi) = (-1)^{i+1} \delta_2 + D(Wi)
\]

\[
\sum_{n=0}^{M} a(n) \cos(w_i n) = \frac{(-1)^{i+1} \delta_2}{W(w_i)} + D(W_i)
\]

\( i = 1, \ldots, M+2 \)

\[ \rightarrow \text{linear in } a(n) \text{ and in } \delta_2 \text{; i.e. linear system } \]

4 equations
\[ i = 1 \]
\[ a(0) \cos(w_1) + a(1) \cos(w_2) + a(2) \cos(2w_1) + \ldots + a(M) \cos(Mw_1) \]

\[ = (-1)^{i+1} \frac{S_2}{W(w_1)} + D(w_1) \]

\[ i = 2 \]
\[ a(0) \cos(w_2) + a(1) \cos(w_2) + \ldots + \ldots + \ldots + \ldots \]

\[ i = M+2 \]
\[ \ldots \]

\[ \underline{\text{Eqn is linear in both } a()} \text{ and in } S_2 \]

\[ M+2 \text{ eqns } + \text{ M+2 unknowns} \Rightarrow \text{ get answer} \]
if \( w_i \) known

entries of this matrix is also entirely known

\[ A \mathbf{x} = \mathbf{b} \]
Remez Exchange Algorithm.

Given $w_i$, $S_2$.

Parke and McCulloch showed: if $w_i$ are known, $S_2$ is given by the following expression.

$$S_2 = \sum_{k=1}^{M+2} b_k \frac{D(w_k)}{W(w_k)}$$

$$b_k = \frac{1}{M+2} \sum_{i=1}^{M+2} \frac{1}{\cos \omega_k - \cos \omega_i}$$

$$b_k = \frac{1}{M+2} \sum_{i \neq k} \frac{(-1)^{k+1}}{W(w_k)}$$
Empirical Studies

Approx length of filter $\leq 1 - \frac{10 \log_{10} (6.62) - 13}{2.35} \Delta f$.

$\Delta f = W_s - W_p$

Kaiser window length $\approx 1 + \frac{A-8}{2.2 \Delta f}$, $A = 20 \log_{10} \delta$

$\hat{b}_k$, $W_p = 0.417$  \hspace{1cm} $W_s = 0.691$  \hspace{1cm} $\xi_1 = 0.01$  \hspace{1cm} $\xi_2 = 0.001$

Optimum filter

Kaiser window

$\# \psi T_{\gamma} = 2M + 1 = 27$

$\# \psi T_{\hat{b}} = 38$

Sim 7.43
IIR Filter Design

Interested in IIR filters that have rational transfer functions.

\[ H(z) = \frac{P(z)}{Q(z)} \]

\[ P(z) = \sum_{n=0}^{N} a(n) z^{-n} \]

\[ Q(z) = \sum_{s} b(n) z^{-s} \]

Filter design is to find \( a(n) \) and \( b(n) \).
1. Given set of Discrete Time Digital Filter Spec
2. Transform specs from Discrete Time to Continuous-time
   \[ \mathbb{Z} \rightarrow \mathbb{S} \]
3. Design Continuous-time \( \mathbb{I} \rightarrow \mathbb{R} \) (analog)

4. \( H_a(s) \) \[ \rightarrow \] \( H(z) \)
   Continuous-time \[ \rightarrow \] Discrete-time

\[ \mathbb{S} \rightarrow \mathbb{Z} \]
1. Butterworth
   monotonic in passband and stopband

Chebyshev
   ripple in passband or stopband
   but not in both

Continuous time IIR filters

\[ \text{Phase} \]

\[ \rightarrow \text{Ripple in passband} \]
Cholesky

Elliptichilten

H(a(z))

Ripple in stopband.
Desirable. Properties of Transformation

1. Causal and stable analog filter to get transformed into causal and stable discrete time filters.
   \[ H_a(s) \xrightarrow{x-form} H(z) \]
   Causal + Stable

2. \( j\omega \) axis in \( s \)-plane to get transformed onto \( \gamma \) axis in \( z \)-plane
   \( e^{j\omega} \) circle in \( z \)-plane

Diagram:
- Causal + Stable
- \( j\omega \) axis in \( s \)-plane
- Transformation to \( \gamma \) axis in \( z \)-plane
Need this to translate spaces from discrete to analog:

\[ e^{j\omega} \rightarrow s \in \mathbb{C} \]

\[ \text{z-plane} \rightarrow \text{s-plane} \]

3. Rational H(s) \rightarrow Rational H(z). Discrete analogy


\[ s = \frac{s}{s^2 + \omega^2} \]

\[ s = \sqrt{2} \]
Impulse Invariant Transformation

\[ H_a(s) \rightarrow h_a(t) \rightarrow h(n) = \left[ h_a(t) \right]_{t=nT} \rightarrow H(z) \]

Does this satisfy all 3 desirable properties?
Does causal + stable \( H(s) \) Tremain with causal + stable \( N(z) \)?

\[
H(s) = \sum \frac{A_k}{s - s_k}
\]

Causality + stability \( \iff \) poles have to be in left half plane.

\[ \Rightarrow \text{Re}[s_k] < 0 \]

\[ h(t) = \sum A_k e^{s_k t} \]

Exploit causality.
\[ h(n) = \left( h_a(t) \right)_{t=nT} = \sum_{k} A_k e^{s_k T} u(n) \]

\[ H(z) = \sum_{k} A_k \frac{1}{1 - e^{-s_k T} z^{-1}} \]

Q. Is the filter stable?

Is poles inside unit circle?

\[ | e^{-s_k T} | < 1 \]
\[ \Re \{ s_k \} < 0 \implies |e^{sk}T| < 1 \iff \text{poles are inside unit circle.} \]

\[
\left| \frac{e^{sk}T}{e^{sk}T \circ e} \right| = \left| e^{sk}T \right| < 1
\]

\[ \Re \{ s_k \} < 0 \to \text{Show that causal and stable H(s)} \to \text{Causal + stable H(s)} \]

\[ \text{Also show rational H(s)} \to \text{Rational H(s)} \]
(1) Denote axis in s plane \( \rightarrow e^{j\omega} \) circle in \( \mathbb{Z} \) plane.

\[ h(n) = \left[ h_a(t) \right]_{t=nT} \]

\[ H(\omega) = \frac{1}{T} \sum_{k=-\frac{K}{2}}^{\frac{K}{2}-1} H_a \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \]