Nov 4, 05

Algorithms for Optimal Filter Design:

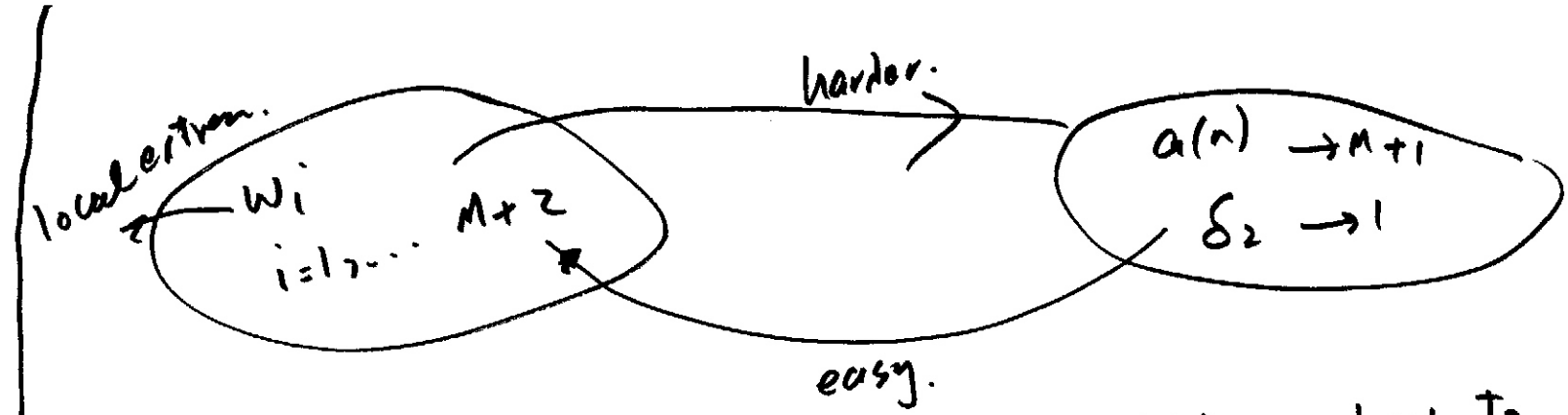
Problem B: Given ω_s, ω_p, $K = \frac{\delta_1}{\delta_2}$, M

find $\underline{\underline{a(n)}}$ to minimize δ_2

$$G(\omega) = \sum_{n=0}^{M} a(n) \cos(\omega n)$$

Alt Thm:

$$\begin{cases} E(\omega_i) = \pm \delta_2 & i = 1, \ldots, M+2 \\ E(\omega_i) = -E(\omega_{i+1}) & i = 1, \ldots, M+1 \\ W(\omega_i)\left[G(\omega_i) - D(\omega_i)\right] = (-1)^{i+1} \delta_2 . \end{cases}$$



"harder" show if we know w_i, how to find $a(n)$ and δ_2?

$$\left.\begin{aligned} G(w_i) &= \sum_{n=0}^{M} a(n) \cos(w_i n) \\ G(w_i) &= \frac{(-1)^{i+1}\delta_2}{W(w_i)} + D(w_i) \end{aligned}\right\} \Longrightarrow$$

$$\sum_{n=0}^{M} a(n) \cos(w_i n) = \frac{(-1)^{i+1}\delta_2}{W(w_i)} + D(w_i) \qquad i=1,\dots M+2$$

linear in $a(n)$ and in δ_2; ie linear system of equations

$i=1$

$$a(0)\cos(\omega_1 \cdot 0) + a(1)\cos(\omega_1) + a(2)\cos(2\omega_1) + \dots a(M)\cos(M\omega_1) = \frac{(-1)^{1+1}\delta_2}{W(\omega_1)} + D(\omega_1)$$

$\vdots$

$i=2$ $\quad a(0)\cos(\omega_2 \cdot 0) + a(1)\cos(\omega_2) \dots$

$\vdots$

$i = M+2$

---

eqn is linear in both $a()$ and in δ_2

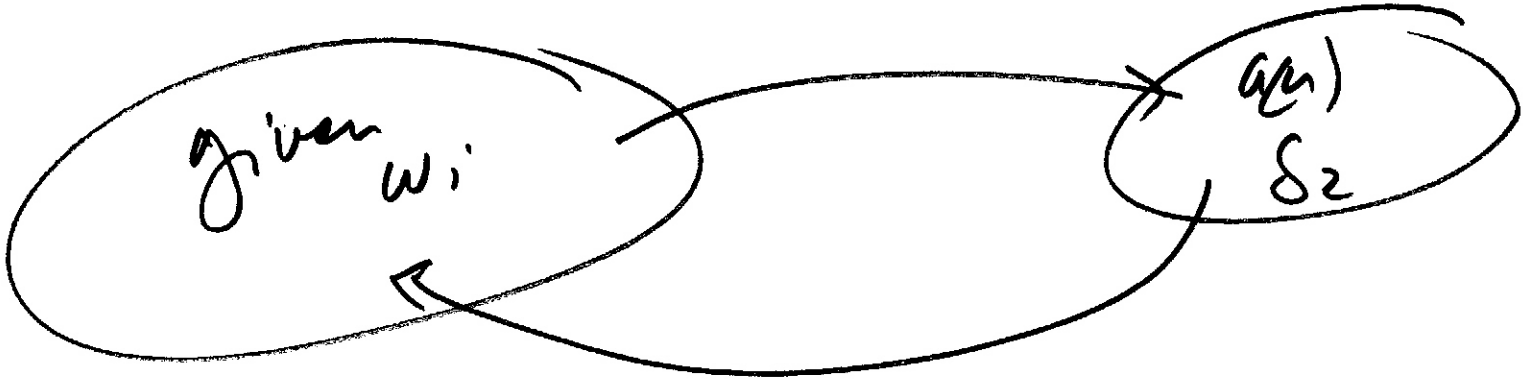
$M+2$ eqns + $M+2$ unknown $\Longrightarrow$ get answer

$$
\underbrace{\begin{bmatrix}
\cos(w_1 \cdot 0) & \cos w_1 & \cos 2w_1 & \cdots & \cos w_1 M \\
\cos 0 & \cos w_2 & \cos 2w_2 & \cdots & \cos M w_2 \\
\cos 0 & \cos w_3 & \cos 2w_3 & \cdots & \cos M w_3 \\
\vdots & & & & \\
\cos 0 & \cos(w_{M+2}) & \cos(2 w_{M+2}) & \cdots & \cos M w_{M+2}
\end{bmatrix}}_{\text{if } w_i \text{ known entries in this matrix is also entirely known}}
\underbrace{\begin{bmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \\ a(M) \end{bmatrix}}_{\text{unknown}}
\underset{\text{unknown}}{=} \delta_2
\underbrace{\begin{bmatrix} \frac{(-1)^{1+1}}{W(w_1)} \\ \frac{(-1)^{2+1}}{W(w_2)} \\ \vdots \\ \frac{(-1)^{M+2+1}}{W(w_{M+2})} \end{bmatrix}}_{\text{known}}
+
\underbrace{\begin{bmatrix} D(w_1) \\ D(w_2) \\ \vdots \\ D(w_{M+2}) \end{bmatrix}}_{\text{known}}
$$

$$\underline{A\vec{x} = \vec{b}}$$

$$
\begin{bmatrix}
\cos 0 & \cos w_1 & \cdots & \cdots & \cos M w_1 & -\dfrac{(-1)^{1+1}}{W(w_1)} \\
\cos 0 & \cos w_2 & \cdots & \cdots & \cdots & \\
\cos 0 & \cos w_3 & \cdots & \cdots & \cos M w_3 & -\dfrac{(-1)^{2+1}}{W(w_2)} \\
\vdots & & & & & \\
\vdots & & & & & \\
\vdots & & & & & \\
\cos 0 & \cos w_{M+2} & \cdots & \cdots & \cdots & -\dfrac{(-1)^{M+2+1}}{W(w_{M+2})}
\end{bmatrix}
\begin{bmatrix} a(0) \\ a(1) \\ \vdots \\ a(M) \\ \delta_2 \end{bmatrix}
=
\begin{bmatrix} D(w_1) \\ D(w_2) \\ \vdots \\ D(w_{M+2}) \end{bmatrix}
$$

Remez Exchange Algorithm.



Parks + McClellan:

showed: if ω_i are known, δ_2 is given by the following expression

$$\delta_2 = \frac{\sum_{k=1}^{M+2} b_k D(\omega_k)}{\sum_{k=1}^{M+2} \frac{b_k (-1)^{k+1}}{W(\omega_k)}}$$

$$b_k = \prod_{\substack{i=1 \\ i \neq k}}^{M+2} \frac{1}{\cos\omega_k - \cos\omega_i}$$

Empirical Studies

Appro length of filter $\simeq 1 - \dfrac{10 \log_{10}(\delta_1 \delta_2) - 13}{2.3 \, \Delta\omega}$.

$\Delta\omega = \omega_s - \omega_p$.

Kaiser: window length $\simeq 1 + \dfrac{A - 8}{2.2 \Delta\omega}$ $\quad A = -20 \log_{10} \delta$

Ex $\omega_p = 0.4\pi$ $\quad \omega_s = 0.6\pi$ $\quad \delta_1 = 0.01$, $\delta_2 = 0.001$

optimum filter $\longrightarrow$ # of taps $= 2M+1 = 27$

Kaise Window $\longrightarrow$ # of tap $\simeq 38$

Show 7.43

IIR Filter Design

- Interested in IIR filters that have rational transfer fns.

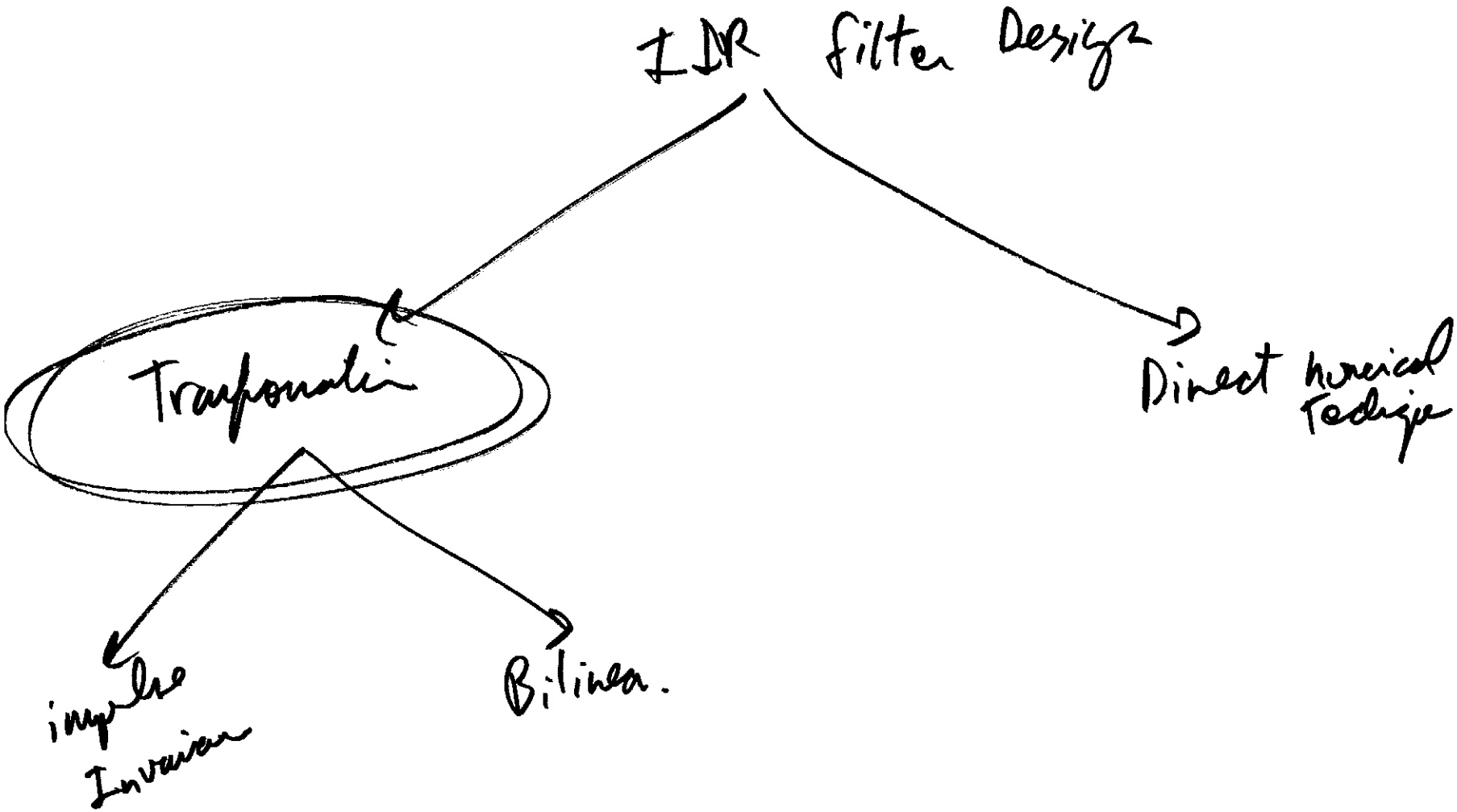
$$H(z) = \frac{P(z)}{Q(z)}$$

$P(z)$, $Q(z)$ → polynomials in z.

$$P(z) = \sum_{n=0}^{N} \underline{a(n)}\, z^{n}$$

$$Q(z) = \sum_{n} \underline{b(n)}\, z^{n}$$

→ Filter Design is to find $a(n)$ and $b(n)$


IIR filter Design
Transformation
Direct numerical Technique
impulse Invariance
Bilinear.

Transformation

1. Given set of Discrete Time Digital Filter spec
2. Transform specs from Discrete Time to Continuous time.
   $z \longrightarrow s$.
3. Design Continuous time IIR (analog).
   $H_a(s)$
   analog.
4. $H_a(s) \xrightarrow{\text{Transform}} H(z)$

   continuous time $\longrightarrow$ Discrete time.

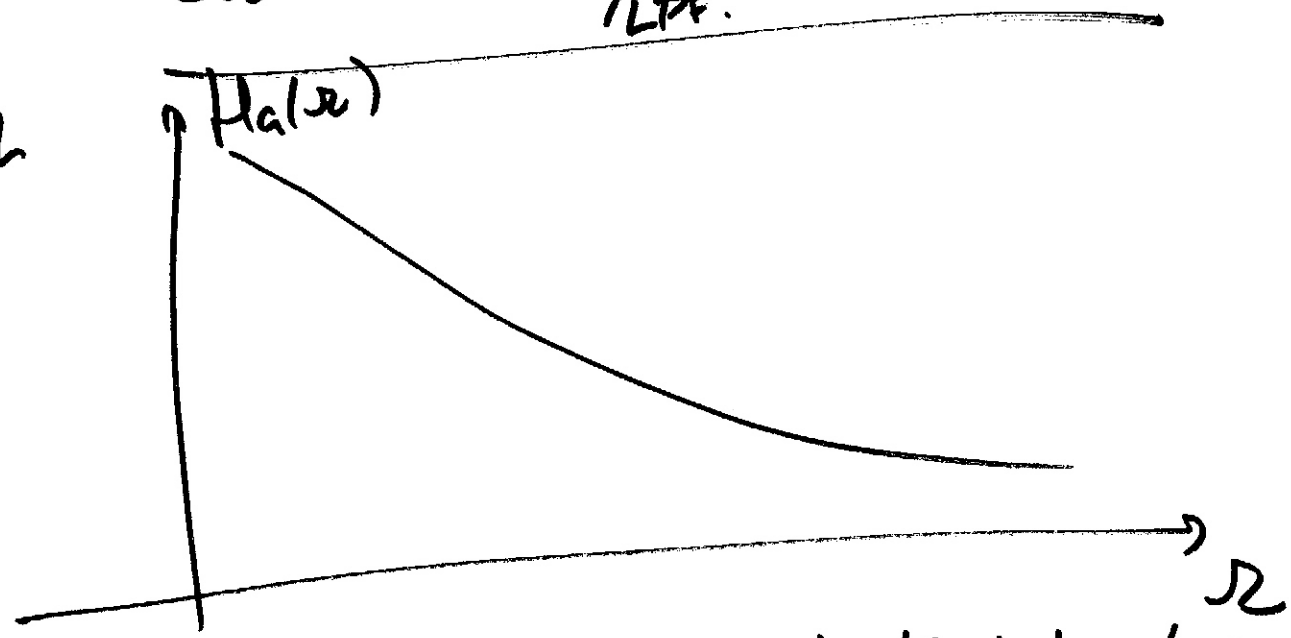
   $s \longrightarrow z$

Continuous time IIR filters

LPF.

1. Butterworth

monotonic in passband + stopband

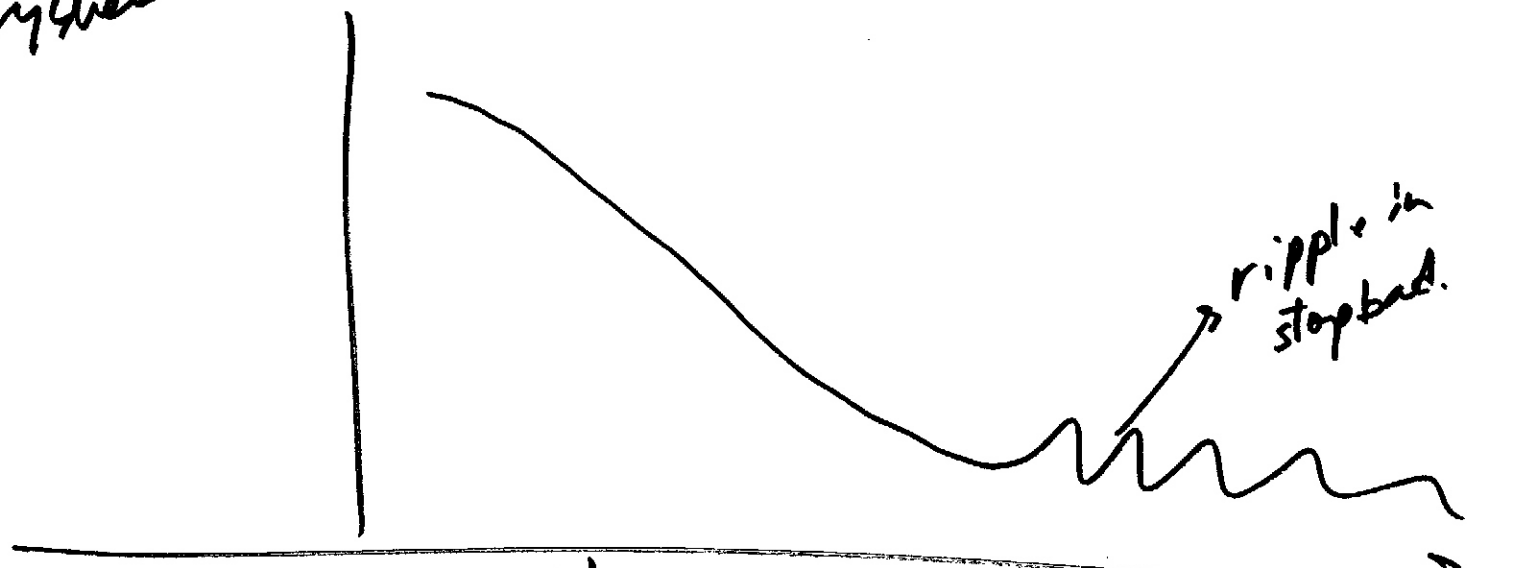


Chebychef

Chebyshev

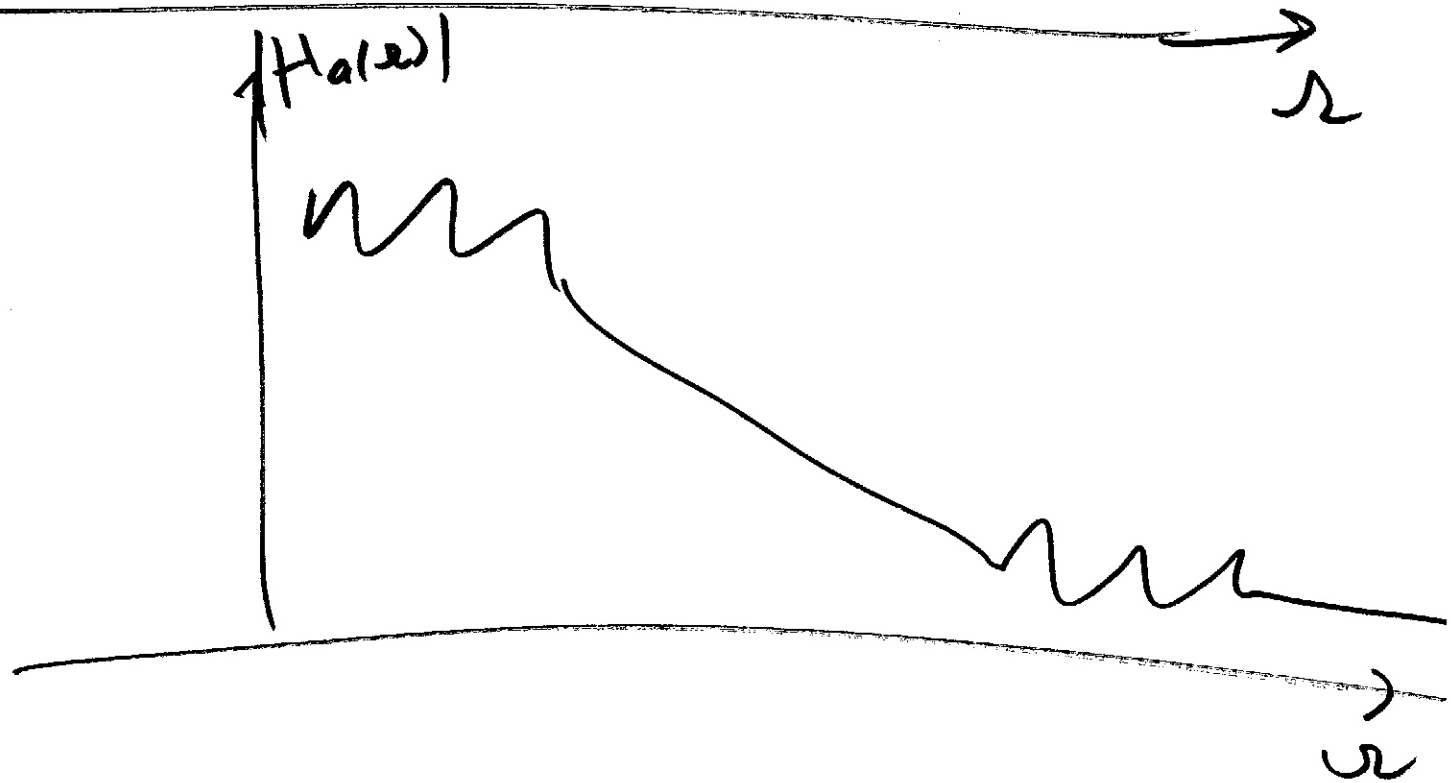
ripple in passband or in stopband but not in both

Chebyshev:



Elliptic filter

ripple in passband and in stopband

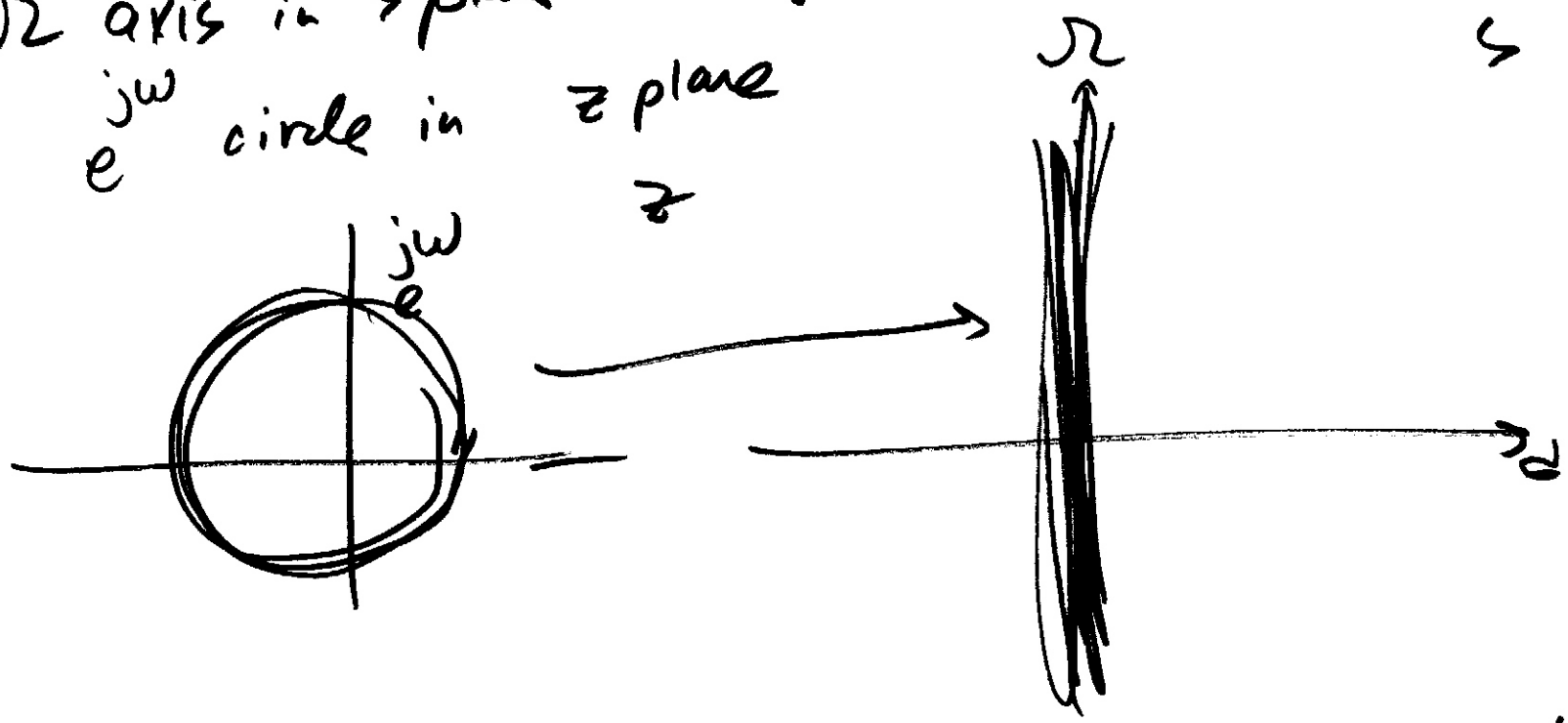


Desirable Properties of Transformation

① Causal + stable analog filter to get transformed into causal + stable discrete time filter.

$$\underset{\text{Causal + stable}}{H_a(s)} \xrightarrow{\text{x-form}} \underset{\text{causal + stable}}{H(z)}$$

② $j\Omega$ axis in s plane to get transformed onto $e^{j\omega}$ circle in z plane



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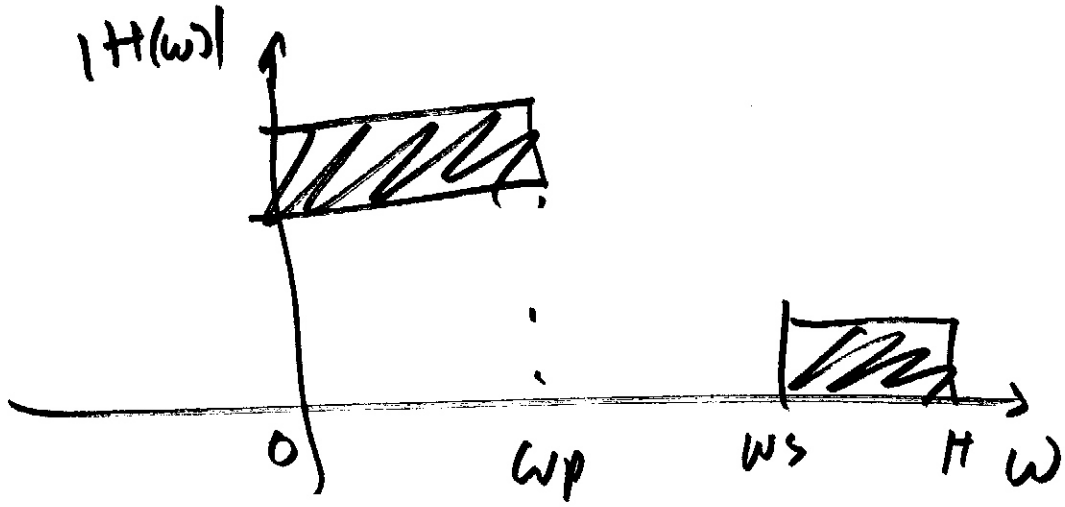

Need This To Translate specs from

Discrete $\longrightarrow$ Analog.

$e^{j\omega}$ $\longrightarrow$ $j\Omega$

zplane. splane.

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(3) Rational $H_a(s)$ $\longrightarrow$ Rational $H(z)$.

analog Discrete

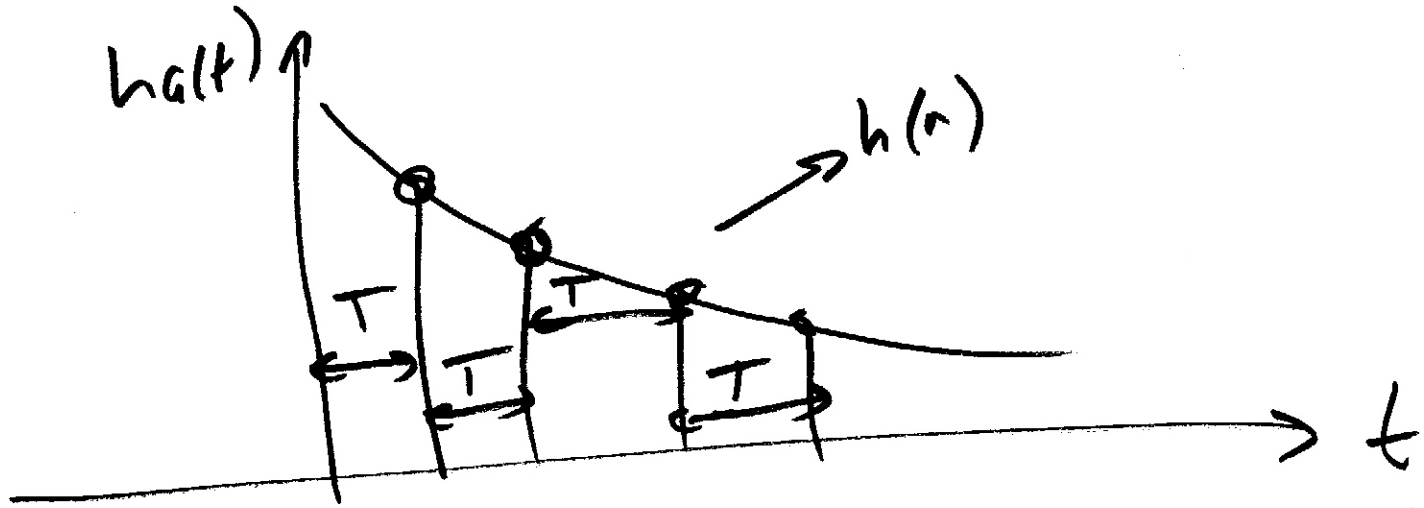
why? Implementation using Diff. eqns.

$z = \sqrt{s}$

$s = \sqrt{z}$

Impulse Invariance Transformation

$$H_a(s) \rightarrow h_a(t) \longrightarrow h(n) = \left[h_a(t)\right]_{t=nT} \rightarrow H(z)$$



Does this satisfy all 3 Desirable properties?

① Does causal + stable $H_a(s)$ Transform into causal + stable $H(z)$?

$$H_a(s) = \sum_k \frac{A_k}{s - s_k}$$

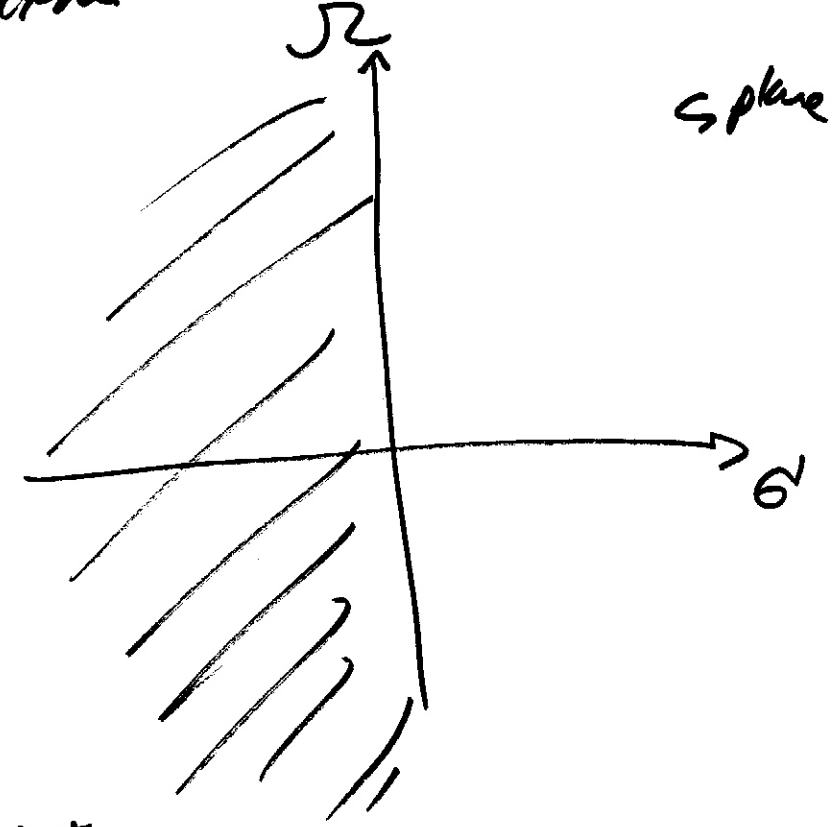
causality + stability $\Longrightarrow$ poles have to be in left half plane.

$$\Longrightarrow Re[s_k] < 0$$

$$h_a(t) = \sum_k A_k e^{s_k t} u(t)$$

exploit causality

assume causal + stable



$\downarrow$ Transformation.

$$h(n) = \left[h_a(t) \right]_{t=nT} = \sum_k A_k \, e^{s_k n T} \, \underline{u(n)}$$

This is causal. ✓

$\downarrow$

$$H(z) = \sum_k A_k \frac{1}{1 - e^{s_k T} z^{-1}}$$

<u>Q</u> is this filter stable?

is poles inside unit circle?

$$\left| e^{s_k T} \right| < 1 \; ??$$

$$\mathrm{Re}[s_k] < 0 \Longrightarrow \left| e^{s_k T} \right| < 1 \Longrightarrow$$

poles are inside unit circle.

$$\left| e^{s_k T} \right| = \left| e^{\mathrm{Re}[s_k] T} \cdot e^{j \mathrm{Im}[s_k] T} \right|$$

$$= \left| e^{\mathrm{Re}[s_k] T} \right| < 1$$

$\mathrm{Re}[s_k] < 0$

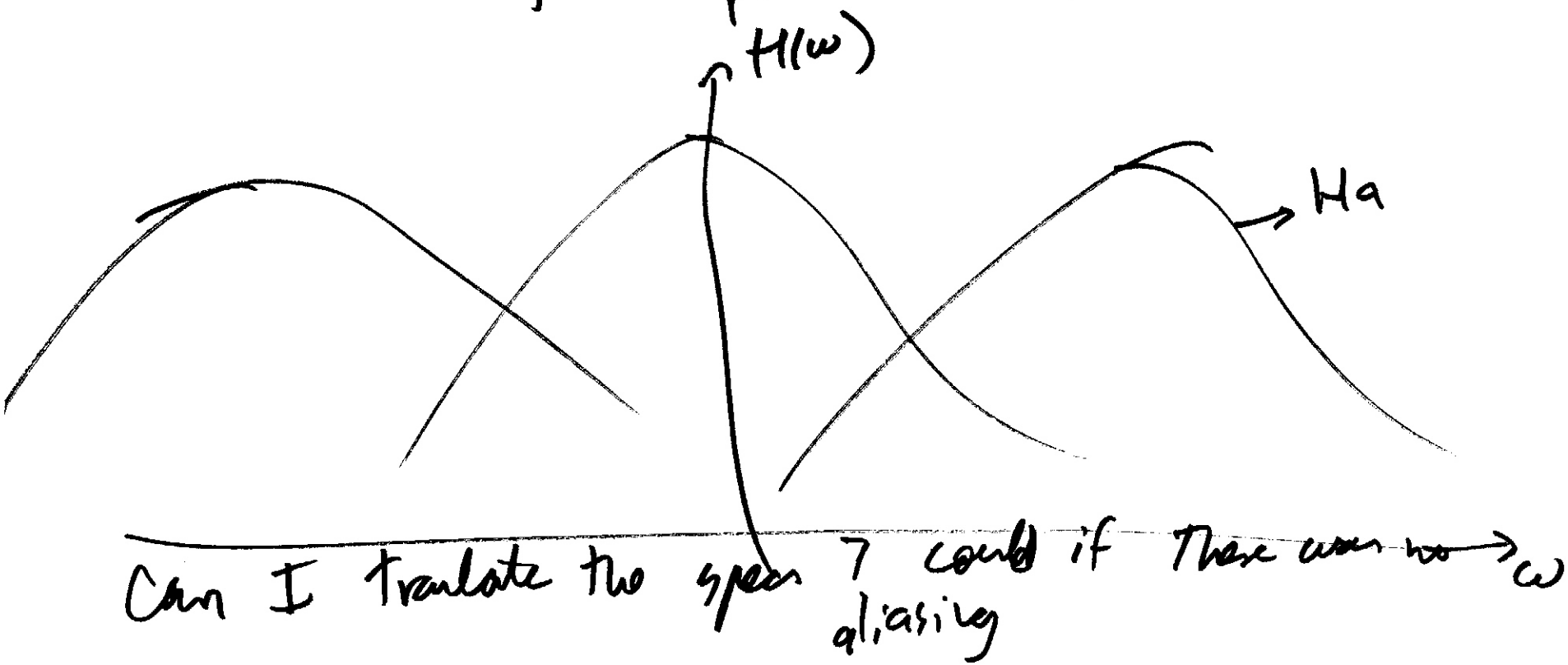
→ Showed Causal + stable $H_a(s)$ ⟶ Causal + stable $H(z)$

→ Also showed Rational $H_a(s)$ ⟶ Rational $H(z)$

① Map $j\Omega$ axis in s plane $\longrightarrow$ $e^{j\omega}$ circle in z plane?

$$h(n) = \left[h_a(t)\right]_{t=nT}$$

$$H(\omega) = \frac{1}{T} \sum_r H_a\left(\frac{\omega}{T} - \frac{2\pi r}{T}\right)$$



Can I translate the spectra? could if there was no