Impulse Invariant Transformation

\[ H_0(s) \rightarrow h_0(t) \rightarrow h(n) = \left[ h_a(t) \right]_{t=nT} \rightarrow H(z) \]

Does this satisfy all 3 desirable properties?
6. Does causal + stable \( H(s) \) transfer into causal + stable \( N(s) \)?

\[
H(s) = \sum \frac{A_k}{s - s_k}
\]

Causality + stability \( \Rightarrow \) poles have to be in Left half plane.

\[\Rightarrow \text{Re} [s_k] < 0\]

\[
\text{h}(t) = \sum \text{e}^{s_k t} u(t)
\]

\[1 \quad \text{explore causality}\]
\[ h(n) = (h_a(t))_{t=nT} = \sum_{k} A_k e^{s_k nT} u(n) \]

\[ H(z) = \sum_{k} A_k \frac{1}{1 - e^{-s_k T}} \]

Q: Is the filter stable?

Is poles inside unit circle?

\[ |e^{-s_k T}| < 1 \] ?
\[ \text{Re}[s_k] < 0 \implies |e^{s_k T}| < 1 \implies \] poles are inside unit circle.

\[
\left| e^{s_k T} \right| = \left| e^{Re[s_k] T} \right| < 1
\]

\[ \implies \text{Showcase} \quad \text{Rational H}(s) \implies \text{Rational H}(z) \]

\[ \implies \text{Also showcase} \quad \text{Rational H}(s) \implies \text{Rational H}(z) \]
\[ h(n) = [h_a(t)]_{t=nT} \]

\[ H(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \]

Can I translate the space \( T \) could if there was no aliasing?
Example of "Impulse Invariant Transformation"

IIR Filter Design

1. Given specifications
   D.T. Domain

\[ 0 \leq |H(\omega)| \leq 1 \quad 0 \leq |\omega| \leq 0.2 \pi \]
\[ |H(\omega)| < 0.172 \quad 0.3 \pi < |\omega| < \pi \]
Step (c) Transform D.T. specs into C.T. specs.

Choose $T = 1$

$H_d(z)$

<table>
<thead>
<tr>
<th>z</th>
<th>-1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0.89</td>
<td>1</td>
</tr>
</tbody>
</table>

Design C.T. filter that satisfies specs.

Approach: Butterworth filter.
tutorial on butterworth filter of order \( N \):

\[ |H_0(j\omega)|^2 = \frac{1}{1 + \left( \frac{\omega}{\omega_c} \right)^{2N}} \]  

\( \omega_c = \) cut off frequency, \( N = \) order.

To minimize the order of the resulting filter, let us impose the following constraints:

\( \omega_c \) and \( N = \) small.
\[ \frac{1}{\text{Re}} \left( \frac{0.2\eta}{\text{Re}} \right)^2 \frac{1}{1 + \left( \frac{0.3\eta}{\text{Re}} \right)^2} = \frac{1}{1 + \left( \frac{0.2\eta}{\text{Re}} \right)^2} \]

\[ N = \frac{5.88}{0.704} \approx 8.37 \]

To compute \( N \), use known values:

\[ \eta = 0.19 \]

Combine with eqn 1

\[ (0.89)^2 = (0.19)^2 \]

\[ R_e = 0.3 \eta \]

\[ R_e = 0.2 \]

\[ N = 5.88 \approx 8.37 \]
Read \( N = 5.88 \rightarrow \) to \( N = 6 \). To have integer as order.

Must change \( \mathcal{R} \mathcal{C} \rightarrow \mathcal{R} \mathcal{C}' \).

**Question**: Which eqn should I use to re-compute \( \mathcal{R} \mathcal{C}' \) given \( N = 6 \)?

\( \Rightarrow \) Two choices.

1. Plug into eqn \( \mathcal{R} \mathcal{C} \) to recompute \( \mathcal{R} \mathcal{C} \).

2. Plug into eqn \( \mathcal{R} \mathcal{C}' \).
If plug into Eqn 1

Choose this
To minimize
aliasing.

Plug into Eqn 2

Oversatisfy
Stopband.

Oversatisfying
Spec in passband.
\[ (0.89)^2 = \frac{1}{1 + \left( \frac{0.21}{Rc} \right)^{12}} \]

\[ \Rightarrow Rc = 0.7032 \]

Done with Butterworth Analog Filter Design.

\[ Rc = 0.7032 \]

\[ N = 6 \]

\[ |H_a(\omega)|^2 = \frac{1}{1 + \left( \frac{\omega}{0.7032} \right)^{12}} \]

\[ H_a(s) H_a(-s) = \frac{1}{1 + \left( \frac{s}{j \omega c} \right)^{2N}} \]
$S_1, S_6 = 0.182 \pm j\ 0.679$

$S_2, S_5 = -0.497 \pm j\ 0.497$

$S_3, S_4 = -0.679 \pm j\ 0.182$

Angle between poles:
\[
\frac{360}{12} = \frac{360}{2N}
\]
\[
H_0(1) = H(1) = \frac{A_k}{S - S_k} \sum_k H(2) - 1
\]

\[
H_a(s) = \left( s^2 + 0.364s + 0.494 \right) \left( s^2 + 0.945s \right)
\]

\[
H(2) = \frac{2.261 - 0.446 - 1}{1 - 1.245s - 0.25s^2} + \frac{1.8557 - 0.6303s}{1 - 1.8s + 0.72s^2 - 1}
\]

\[
H(1) = \frac{2.891 - 0.446 - 1}{1 - 1.245s - 0.25s^2} + \frac{1.8557 - 0.6303s}{1 - 1.8s + 0.72s^2 - 1}
\]
Bilinear Transformation

\[
H_d(z) = \left[ H_a(s) \right]_{s=\frac{2}{T}} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right)
\]

\[\text{discrete}\]

\[\text{Since } \frac{1 - z^{-1}}{1 + z^{-1}} \text{ is rational then...}\]

Rational Transfer in C.T. \rightarrow \text{Rational D.T. Transfer in } \mathbb{C}_T

\[\text{Does } j\omega \text{ axis get mapped onto } e^{j\omega} \text{ circle? (z plane).}\]

(space)
Let $z = e^{2\pi i/3}$, then

$$s = \frac{2}{1 - e^{2\pi i/3}}$$

$$s = \frac{2}{2 - e^{2\pi i/3}}$$

$$s = \frac{2}{2 - \cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}}$$

$$s = \frac{2}{2 - \frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

$$s = \frac{2}{\frac{3}{2} + i\frac{\sqrt{3}}{2}}$$

$$s = \frac{4}{3 + 2i\sqrt{3}}$$

$$s = \frac{4(3 - 2i\sqrt{3})}{9 + 12}$$

$$s = \frac{4(3 - 2i\sqrt{3})}{21}$$

$$s = \frac{4}{7} \left( \frac{3}{7} - \frac{2\sqrt{3}}{7}i \right)$$

$$s = \frac{12}{7} - \frac{8\sqrt{3}}{7}i$$

2 plane
\[
\begin{align*}
G &= 0 \\
J_z &= \frac{2}{T} \tan \frac{W}{2}
\end{align*}
\]

As \( w \to 0 \):
- \( J_z \to 0 \)
- \( \eta \to \infty \)
- \( \nu \to 0 \to \infty \)
- \( \eta \to -\infty \)
- \( \eta \to 0 \to \infty \)

Q. Does canal stable Hals) \( \to \) Canal + Stall \( H(z) \)?

Need to show LHP in 5 doni
is mapped into \( \text{INSIDE} \) unit circle in 2 doni.

- only need to show this point.
  (since Trajecturi w,r is smooth, cont. diff.)
Pick \( s = -\frac{2}{3T} \)

\[
S = \frac{2}{T} \left( \frac{1 - e^{-sT}}{1 + e^{-sT}} \right) = -\frac{2}{3T}
\]

\( \Rightarrow 2z = \frac{1}{2} \)

\( \Rightarrow \) LHP \( \rightarrow \) inside unit circle.

\( \Rightarrow \) Canal stable \( H(s) \) \( \rightarrow \) Canal stable \( H(z) \)
$n = \frac{2}{T} \tan \frac{\omega}{2}$

Let's use this to translate specs $0.1 \rightarrow \text{analog}$.