Different Realizations of L.C.C.D.E.

Recall a D.E corresponds to many different computational procedures.

Ex: Consider a 2nd order D.E:

\[ y(n) = ay(n-1) + by(n-2) + x(n) \]

3 Realizations of D.E.

- Causal: \( y(n) \leftarrow ay(n-1) + by(n-2) + x(n) \)
  - FRC: \( y(-1) = y(-2) = 0 \)
  - Causal Implication: System \( H(z) \) not outside some circle.

- Anticausal: \( y(n) \leftarrow \frac{1}{b}y(n+2) - \frac{a}{b}y(n+1) - \frac{1}{b}x(n+1) \)
  - FRC: \( y(0) = y(-1) = 0 \)

Sept. 12, 2006
\[ y(0) = y(-1) = 0 \]
if \( x(n) = s(n) \)

\[ y(n-1) \]

anticausal implementation, system

\[ H(z) \text{ ROC inside some circle.} \]

How about I.C.:
\[ y(0) = 0 \]
\[ y(-2) = 0 \]
\[ x(n) \]
\[ y(n) \]
\[ x(n) = s(n) \]
what I.C.? \[ n = 2 \]

\[ y(1) \leftarrow \frac{1}{a} y(n) - \frac{b}{a} y(n-2) - \frac{1}{a} x(n) \]

\[ H(n+1) - \frac{1}{a} y(n+1) - \frac{b}{a} y(n-1) - \frac{1}{a} x(n+1) \]

non-causal.

\[ H(z) \text{ is ring} \]

\[ x(n) \]

\[ y(n) \]

\[ n \]

\[ \frac{1}{180} \text{ degrees} \]

\[ L(\text{degrees}) \]

\( L = (n \mod 2\pi) \)

\( \text{recursive} \)
How to solve LCCDE with FRC on FRC using 2.7.

D.E : corresponds to a causal system.

\[ E_{y(n)} - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = \left( \frac{1}{4} \right)^n u(n) \]

\[ y(n) \leftarrow \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + \left( \frac{1}{4} \right)^n u(n) \]

\[ y(-1) = 0 \]
\[ y(-2) = 0 \]

\[ y(n) \]

\[ y(2) = \frac{1}{1 - \frac{1}{4} z^{-1}} \left( 1 - \frac{1}{2} z^{-1} \right) + \frac{2}{1 - \frac{1}{3} z^{-1}} = \frac{1}{1 - \frac{1}{4} z^{-1}} \]

\[ y(2) = \frac{6}{1 - \frac{1}{2} z^{-1}} + \frac{-8}{1 - \frac{1}{3} z^{-1}} + \frac{3}{1 - \frac{1}{4} z^{-1}} \]
Roc of some circle. To compute 2.2.7. outside

$$y(\omega) = 6 \left( \frac{1}{2} \right) u(\omega) - 8 u(\omega) \left( \frac{1}{3} \right) \frac{1}{4} \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{2} \end{array} \right)$$

$$x = (2)Y \left( \frac{3}{2} + \frac{2}{3} + \frac{3}{2} + \frac{3}{2} \right) = \text{Roc outside}$$

$$y(2) = Y(2) \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{2} \end{array} \right)$$

$$\frac{1}{6} \left[ \frac{1}{6} + \frac{1}{6} - 1 \right] \leq \frac{1}{6} \left[ \frac{1}{6} + \frac{1}{6} - 1 \right]$$

$$h(\omega) = 6 (\omega) \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right)$$

$$H(\omega) = y(2)$$
Realizations of IIR filter with rational transfer function:

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 - \sum_{k=1}^{p} a_k z^{-k}} \]

Assume causality:

\[ y(n) = \sum_{k=1}^{p} a_k y(n-k) + \sum_{k=0}^{q} b_k x(n-k) \]

Assume \( h(n) \) is real, \( a_k, b_k \) are real.
Introduce this notation: flow graph.

1. $X(n) \xrightarrow{c} c \cdot x(n)$  Scaling or mult by $c$.

2. Add 2 signals.

3. Delay $X(n) \xrightarrow{\tau} x(n-1)$

What is the flow graph for this causal system:

$$y(n) \xrightarrow{} a \cdot y(n-1) + x(n)$$
Designing IIR:

1. Specifications
2. Designing $h(n)$
   - Rational Transfer $Q_k, B_k$
3. Realization
4. Implementation
   - ASIC
   - General Purpose SP
   - PC
   - FPGA
4 methods for realization:

1. Direct Form 1
2. Direct Form 2
3. Cascade
4. Parallel

**Direct Form 1**

\[ y(n) = \sum_{k=1}^{p} a_k y(n-k) + \sum_{k=0}^{q} b_k x(n-k) \]

\[ y(n) = \sum_{k=1}^{p} a_k y(n-k) + v(n) \]

\[ y(n) = a_1 y(n-1) + a_2 y(n-2) + \ldots + a_p y(n-p) + v(n) \]
\[ Y(2) = \sum_{k=1}^{p} \sum_{j=2}^{k} x_{(n+j)} + v(2) \]
Direct Form 1.

\[ x(n) \rightarrow \text{numerator of } H, \text{ i.e. } H_1 \rightarrow \begin{cases} \text{denominator} \\ \frac{H_1}{H_2} \end{cases} \rightarrow y(n) \]

Direct Form 2

\[ x(n) \rightarrow H_2 \rightarrow H_1 \rightarrow y(n) \]

Preview
Direct form II.

# of delay

max (p, q).

Lowest possible delay elements to get away with.

Canonical representation.
Realizations of IIR filters with rational transfer function

\[ H(z) = \frac{\sum_{k=0}^{p} b_k z^{-k}}{1 - \sum_{k=1}^{2} a_k z^{-k}} \]

1. Direct Form I
2. \( \frac{v}{u} \)
3. Cascade structure
4. Parallel structure
Fundamental Theorem of Algebra

Any polynomial in one variable can be factored into simple (1st order) terms. Polynomial of degree n, has n real or complex roots. Can be factored into n terms.

Example: 3rd order (degree) polynomial.

\[ P(z) = \alpha z^3 + \beta z^2 + \gamma z + \delta \]

\[ = K (z - z_0)(z - z_1)(z - z_2) \]

where \( z_0, \ z_1, \ z_2 \) are roots of the polynomial.
ILL Conditioned $\rightarrow$ Problem $\rightarrow$ Algorithm

**Problem**

input data $\rightarrow$ Problem $\rightarrow$ Answer soln.

**Ex:** Linear syst of Eqns 2 equation in 2 unknowns

\[
\begin{align*}
5x + 3y &= 2 \\
9x + 20y &= 15
\end{align*}
\]

\[A \cdot x = b\]

**All:** Many algorithms can be used to solve the same problem.

1. Cramer's rule
2. Gaussian elimination with pivot
3. Invert matrix
4. QR Decomposition
5. Gauss-Seidel

16
Problem is ill conditioned. If small perturbation in the input results in large change in the output.

Well-conditioned problem:

\[
\begin{bmatrix}
5 & 3 \\
15 & 20
\end{bmatrix}
\]

Only dependent on the data in the problem itself, not on the Alg used to solve it.

Terrible:

\[
\begin{bmatrix}
15 & 3 \\
16 & 6
\end{bmatrix}
\]

\[\text{det}=0\]

May solve cond. 10, 5.9999

\[\frac{5}{10}\]

A little ill-conditioned,

\[\frac{5}{3}\]

Condition # of a problem: Ratio between relative change in input to relative change in output.

Condition # of problem is large \[\rightarrow\] ill-conditioned

\[\frac{1}{15}\] close to 0

\[\frac{1}{20}\]

Well-conditioned
Problem itself can be inherently either ill conditioned or well conditioned.

If problem is ill-conditioned, you are stuffed.

Condition # of the Alg

relative change in output over relative change in the input. Using that alg to compute input/output.

If problem is well-conditioned, an ill-conditioned alg could give bad results, but a well-conditioned alg give good results.
Statement: The problem of finding the roots of a polynomial is ill-conditioned.

\[ x^3 + \beta x^2 + \gamma x + \delta \]

Small perturbation in \( x, \beta, \gamma, \delta \) change the roots a lot.

Problem is worse for higher degree polynomials.
\[ H(z) = \frac{P(z)}{Q(z)} \]

\[ \Rightarrow \text{Direct Form I & II since they use} \]
\[ \text{coeff of poly in their implementation.} \]

The zeros and poles of the system are very sensitive to round off error.

Finite precision is well respected and

Finite precision in calculation.

Motivates
Consensus Parallel.
Fundamental The of Algebra does not hold in 
2 or 3 or higher # of variables.

\[ p(z_1, z_2) = a z_1^2 + \beta z_2^2 + \gamma z_1 z_2 + \delta \]

\[ + \epsilon z_2 + \zeta z_2 \]

Thm: set of factorable polynomials in 2 or 
more variables is of measure zero 
in the set of - polynomials.

Implication: ① Cannot check stability 
of 2D IIR filters 
easily.

② Cannot come up with 
cascade structure for 
2D IIR.