Sept. 18, 2008

Cascade + Parallel Implementation of IIR filters with Rational Transfer function:

Factoid: If coeffs of polynomial in one variable are real, Then roots are either real or They are complex conjugate.

$\Rightarrow$ If z_0 is root, Then so is z_0^*

- poly of deg 2: $\alpha z^2 + \beta z + \gamma$

$\rightarrow$ Both real

$\rightarrow$ a pair of complex conjugate

- poly of deg 3

$$\alpha z^3 + \beta z^2 + \gamma z + \delta$$

3 real roots

one real. ~~a pair~~ a pair of complex. conjugate roots.

$$K(z-z_0)(z-z_1)(z-z_2)$$

~~Cannot have 2 real roots and one complex~~

- poly of deg 4 : $\alpha z^4 + \beta z^3 + \gamma z^2 + \delta z + \kappa$

4 real roots

2 real roots
a pair of complex conjugate

2 pairs of complex conjugate roots.

$$K(z - z_0)(z - z_1)(z - z_2)(z - z_3)$$

<u>Conclusion</u> : polynomial with real coeff of odd degree always has a real root.

<u>Claim</u>: For a polynomial with real coeff. I can factor it this way:

$$P(z) = \prod_k \underbrace{(1 - c_k z^{-1})}_{\text{real.}} \prod_k \underbrace{(1 - d_k z^{-1})(1 - d_k^* z^{-1})}_{\text{complex}}$$

$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

$$H(z) = A \frac{\prod_k (1 - e_k z^{-1}) \prod_k (1 - f_k z^{-1})(1 - f_k^* z^{-1})}{\prod_k (1 - c_k z^{-1}) \prod_k (1 - d_k z^{-1})(1 - d_k^* z^{-1})}$$

Note: $(1 - d_k z^{-1})(1 - d_k^* z^{-1})$

$$= 1 - 2\,\mathrm{Re}[d_k]\, z^{-1} + |d_k|^2 z^{-2}$$

polynomial with real coeff.

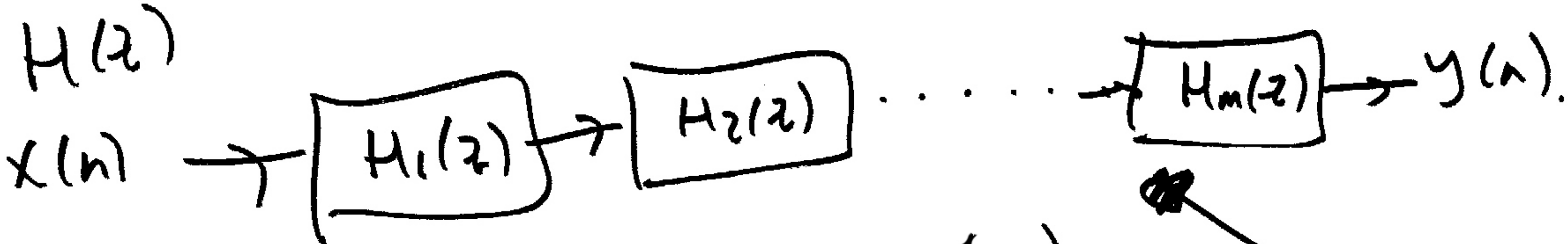
$$= 1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}$$

Generic 2nd order

$$H(z) = A \prod_k H_k(z)$$

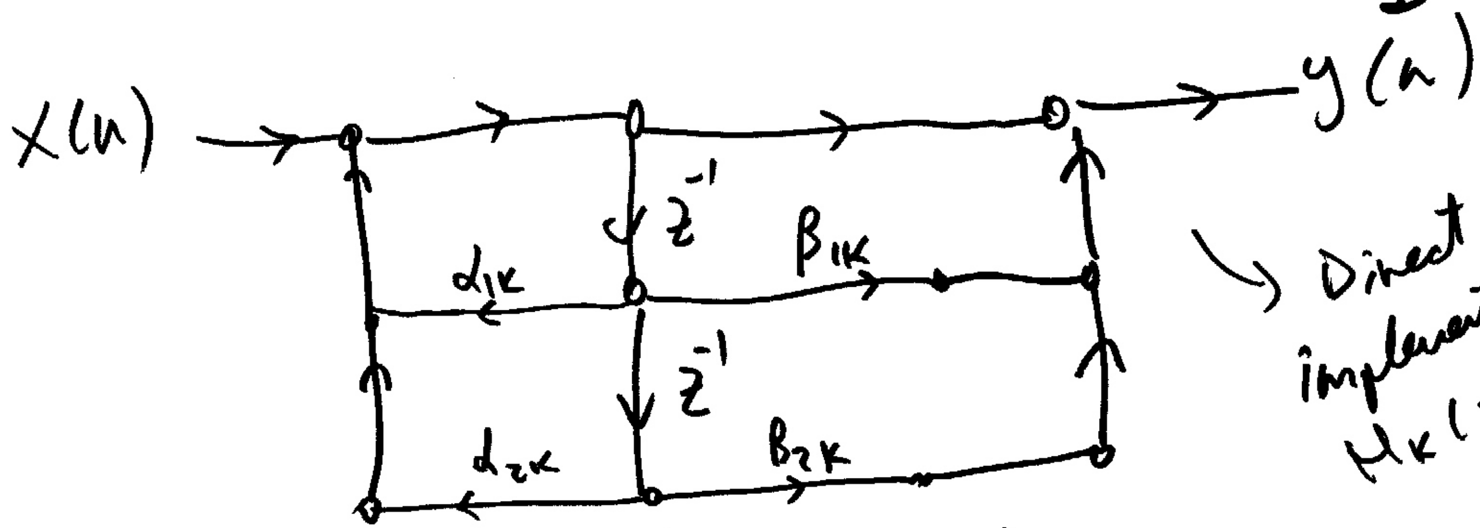
$$H_k(z) = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 - \alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}} = \frac{Y(z)}{X(z)}$$

$\beta_{1k}, \beta_{2k}, \alpha_{1k}, \alpha_{2k}$ are all real.



Cascade Implementation

Q How To implement $H_K(z)$.



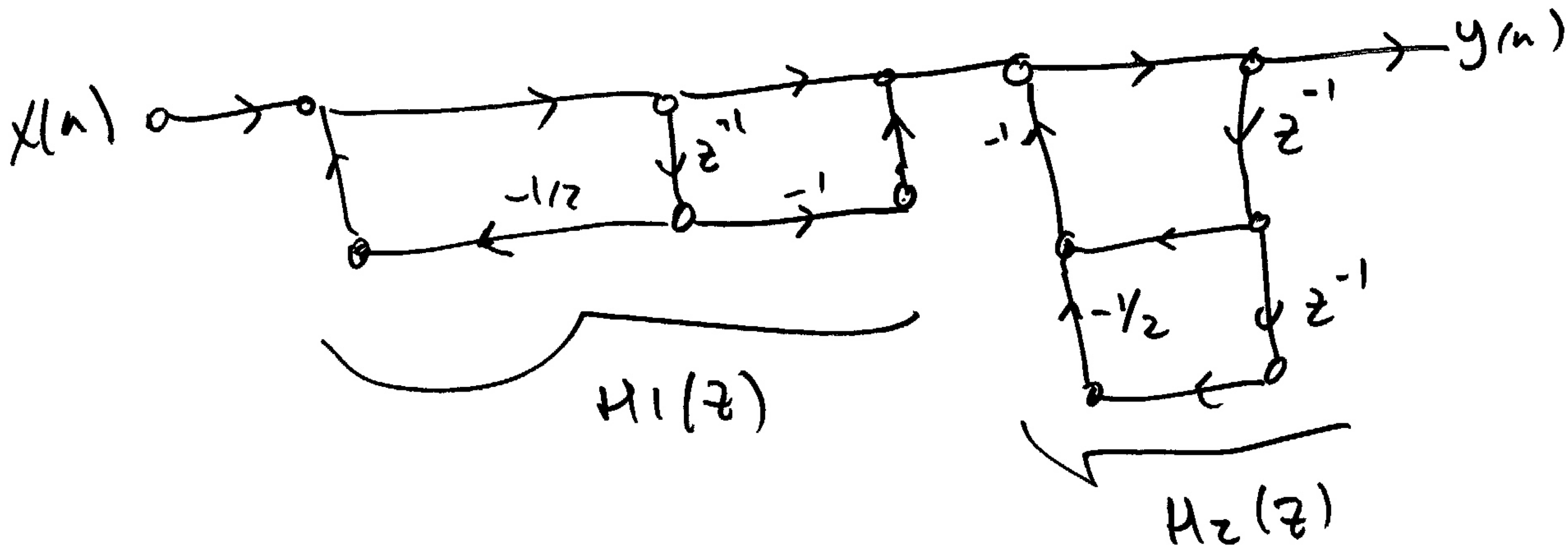
Direct Form II implementation of $H_K(z)$

$$H_K(z) = \frac{1 + \beta_{1K} z^{-1} + \beta_{2K} z^{-2}}{1 - \alpha_{1K} z^{-1} - \alpha_{2K} z^{-2}}$$

Ex $H(z) = \dfrac{1 - z^{-1}}{\left(1 + \frac{1}{2} z^{-1}\right)\left(1 + z^{-1} + \frac{1}{2} z^{-2}\right)}$

$H(z) = H_1(z) \quad H_2(z)$

where $H_1(z) = \dfrac{1 - z^{-1}}{1 + \frac{1}{2} z^{-1}}$ $\quad H_2(z) = \dfrac{1}{1 + z^{-1} + \frac{1}{2} z^{-2}}$



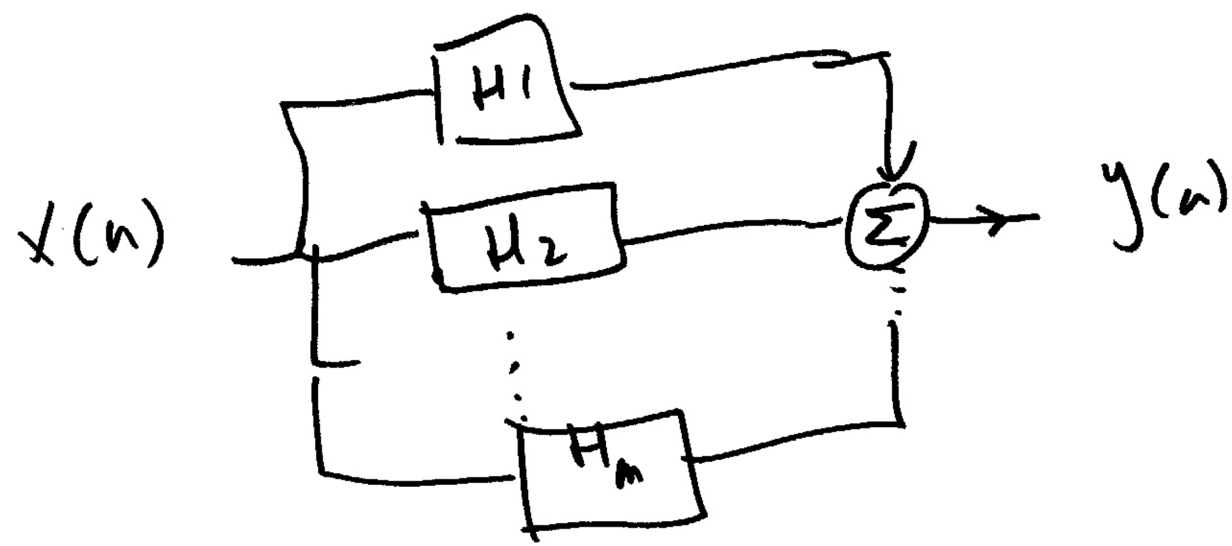
Parallel Implementation

$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 - \sum_{k=1}^{p} a_k z^{-k}} = \sum_k H_k(z)$$

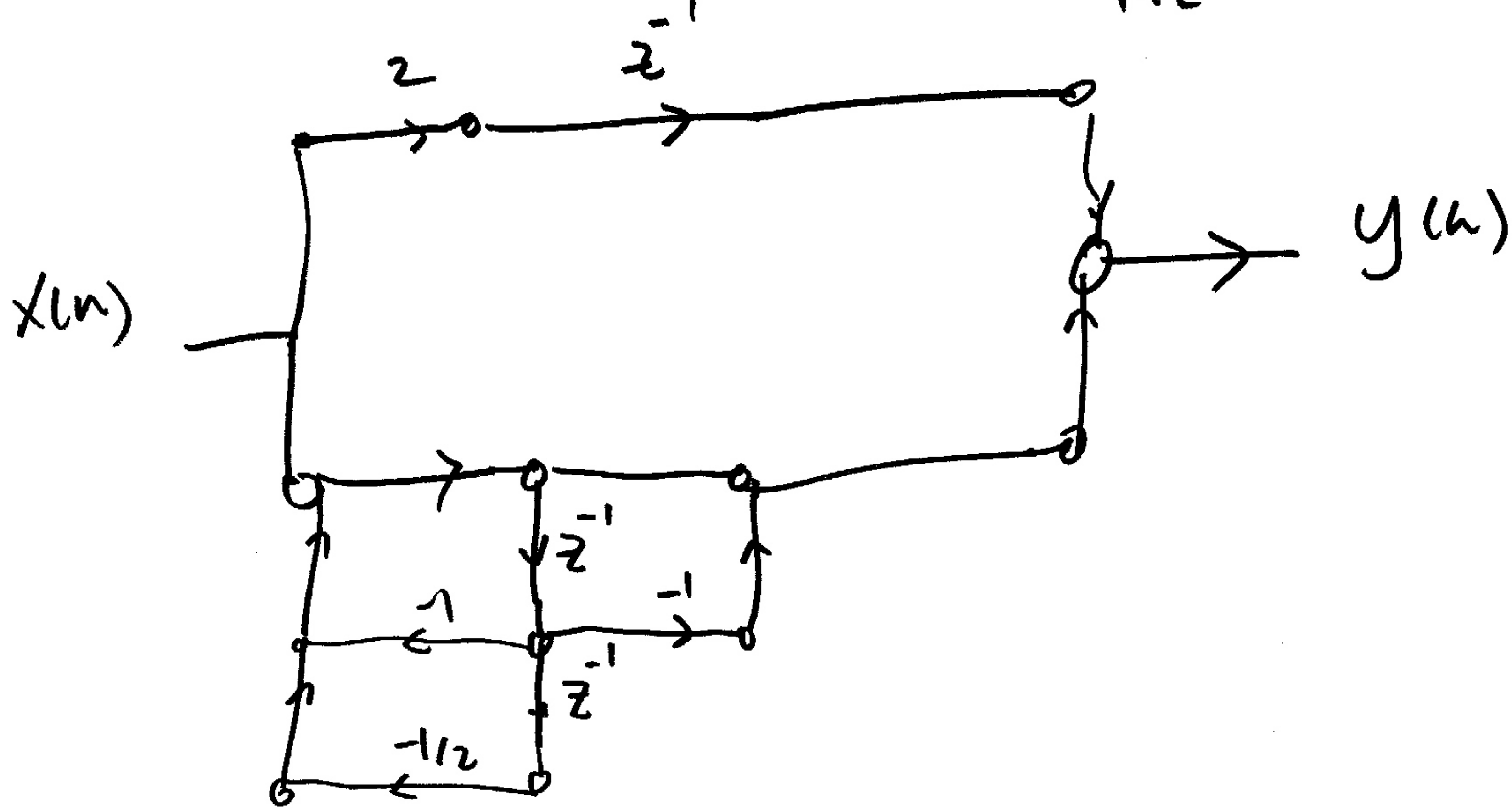
real

$$H(z) = \sum_k A_k z^{-k} + \sum_k \frac{B_k}{1 - g_k z^{-1}} + \sum_k \frac{C_k + D_k z^{-1}}{1 - h_{1k} z^{-1} - h_{2k} z^{-2}}$$

real real

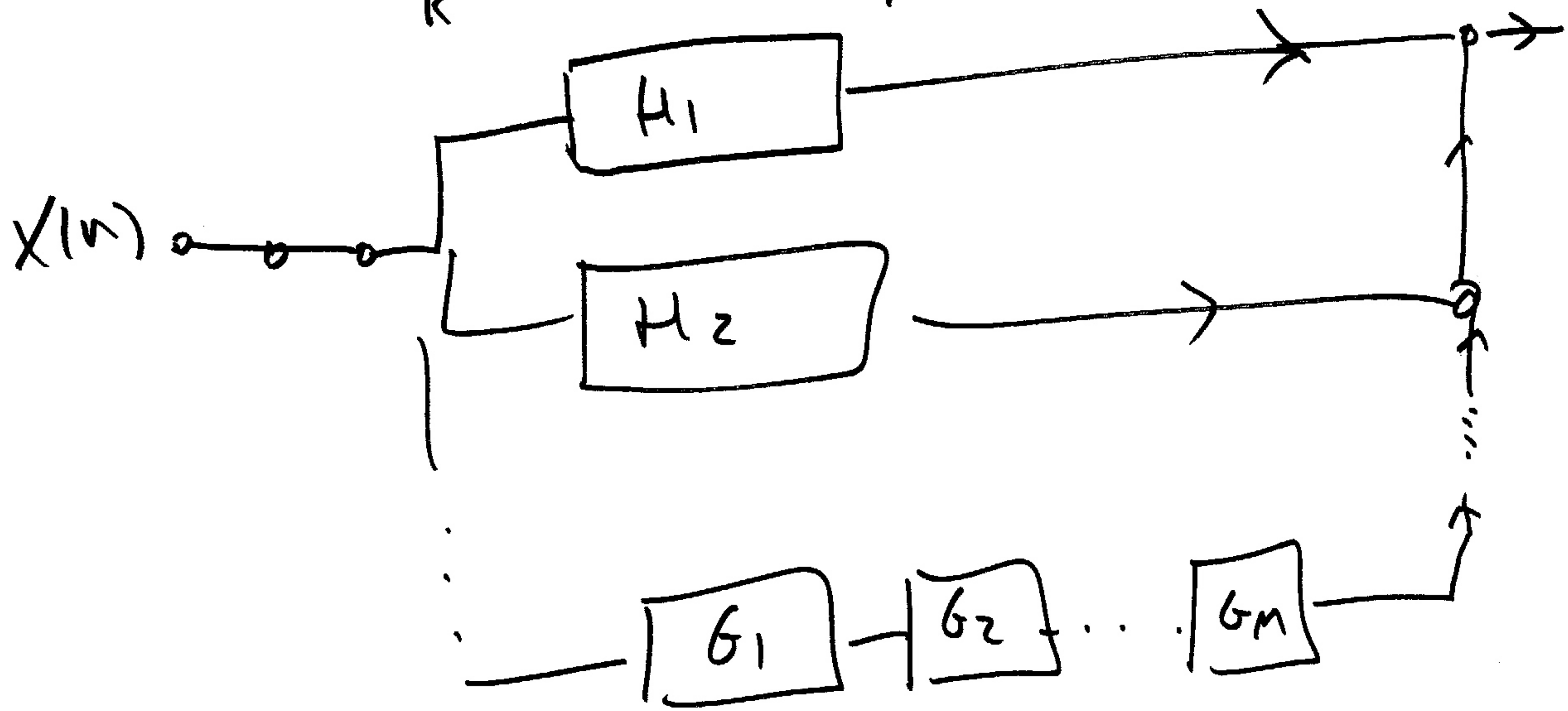


Ex $H(z) = \underbrace{2z^{-1}}_{H_1} + \underbrace{\dfrac{1-z^{-1}}{1+z^{-1}+\frac{1}{2}z^{-2}}}_{H_2}$



Cascade/Parallel structure.

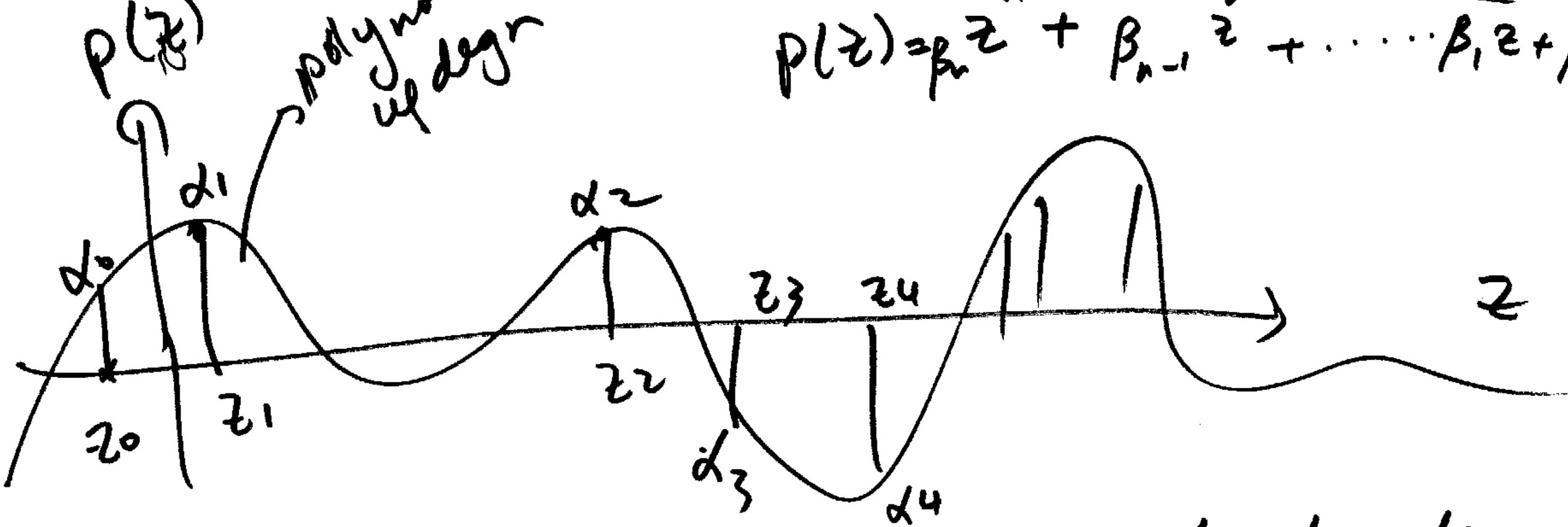
$$H(z) = \sum_k H_k(z) + \prod_k G_k(z)$$



Polynomial Interpolation

To Fundamental Thm of Algebra

$$p(z) = \beta_n z^n + \beta_{n-1} z^{n-1} + \cdots \beta_1 z + \beta_0$$



Any $n+1$ random samples/values of an nth deg deg. polynomial can be used to uniquely reconstruct it.

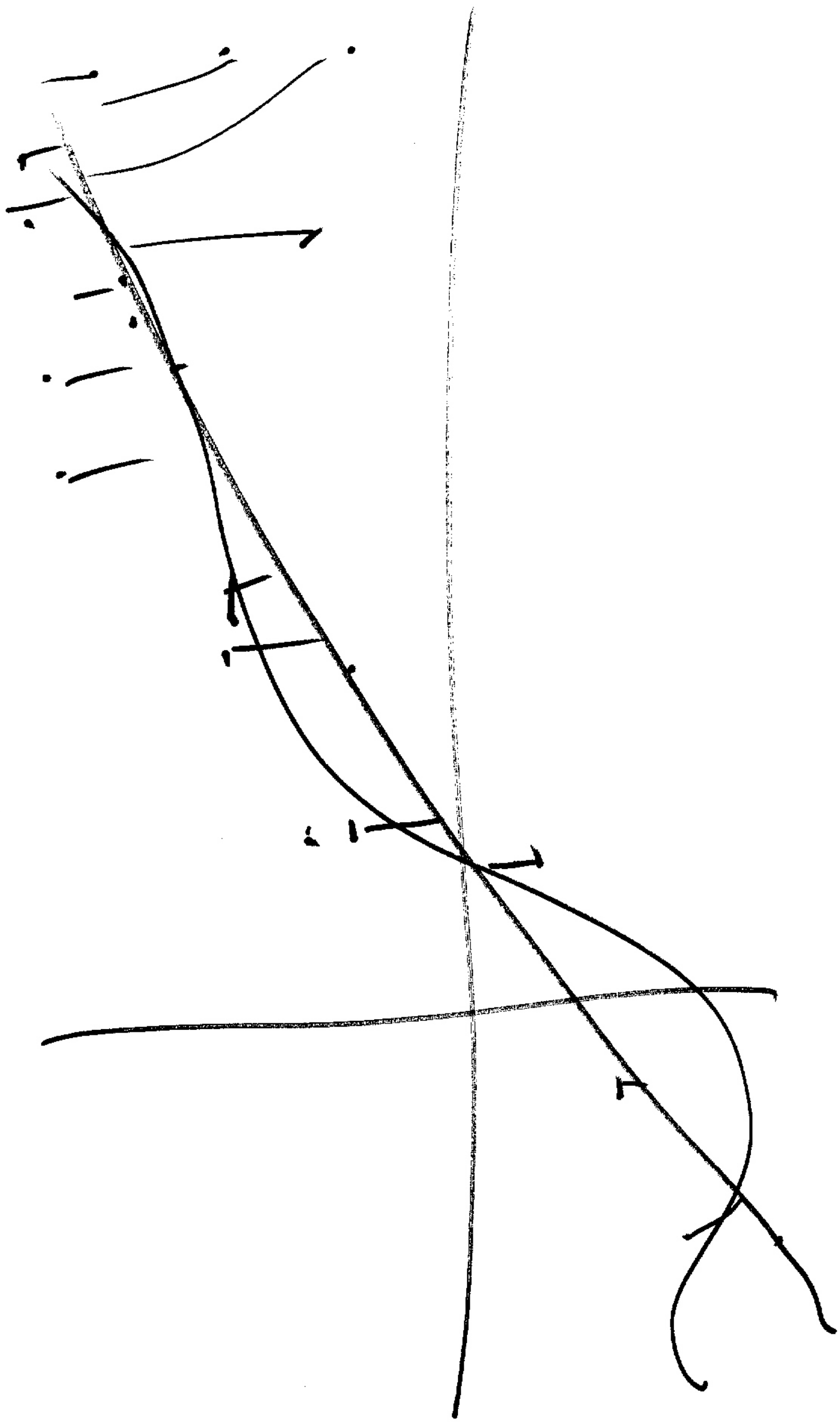
$$
\begin{array}{l}
z_0 \rightarrow \\
z_1 \rightarrow \\
z_2 \rightarrow \\
\vdots \\
z_n \rightarrow
\end{array}
\left|
\begin{array}{l}
\beta_n z_0^n + \beta_{n-1} z_0^{n-1} \cdots\cdots + \beta_1 z_0 + \beta_0 = \alpha_0 \\
\beta_n z_1^n + \beta_{n-1} z_1^{n-1} \cdots\cdots + \beta_1 z_1 + \beta_0 = \alpha_1 \\
\\
\vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \alpha_n
\end{array}
\right.
$$

$$
\underbrace{A}_{(n+1)\times(n+1)} \vec{x} = b
$$

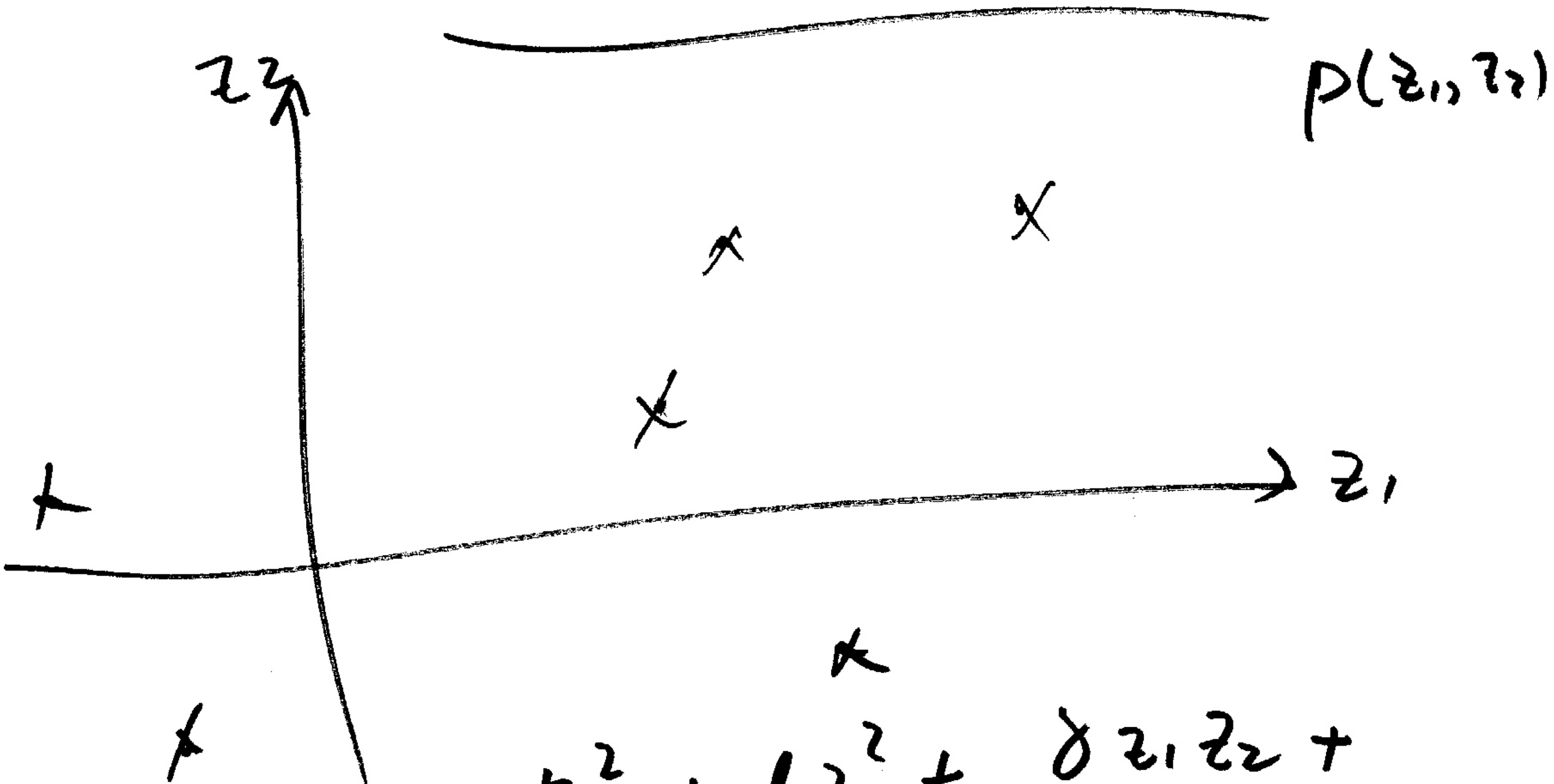
$$
\begin{bmatrix}
z_0^n & z_0^{n-1} & z_0^{n-2} & \cdots\cdots & z_0 & 1 \\
z_1^n & z_1^{n-1} & z_1^{n-2} & & z_1 & 1 \\
z_2^n & z_2^{n-1} & z_2^{n-2} & \cdots\cdots & z_2 & 1 \\
\vdots & & & & & \\
z_n^n & z_n^{n-1} & z_n^{n-2} & \cdots & z_n & 1
\end{bmatrix}
\begin{bmatrix}
\beta_n \\ \beta_{n-1} \\ \vdots \\ \beta_1 \\ \beta_0
\end{bmatrix}
=
\begin{bmatrix}
\alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n
\end{bmatrix}
$$

Van der Monde

non zero determinant.



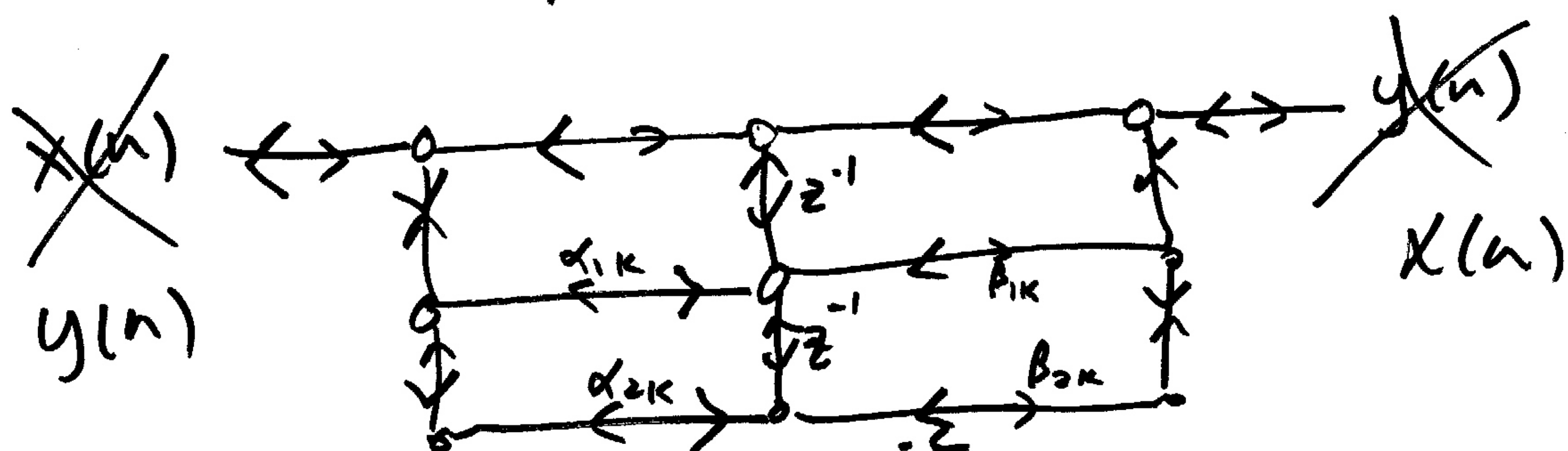
2D Polynomial Interpolation
Does NOT work



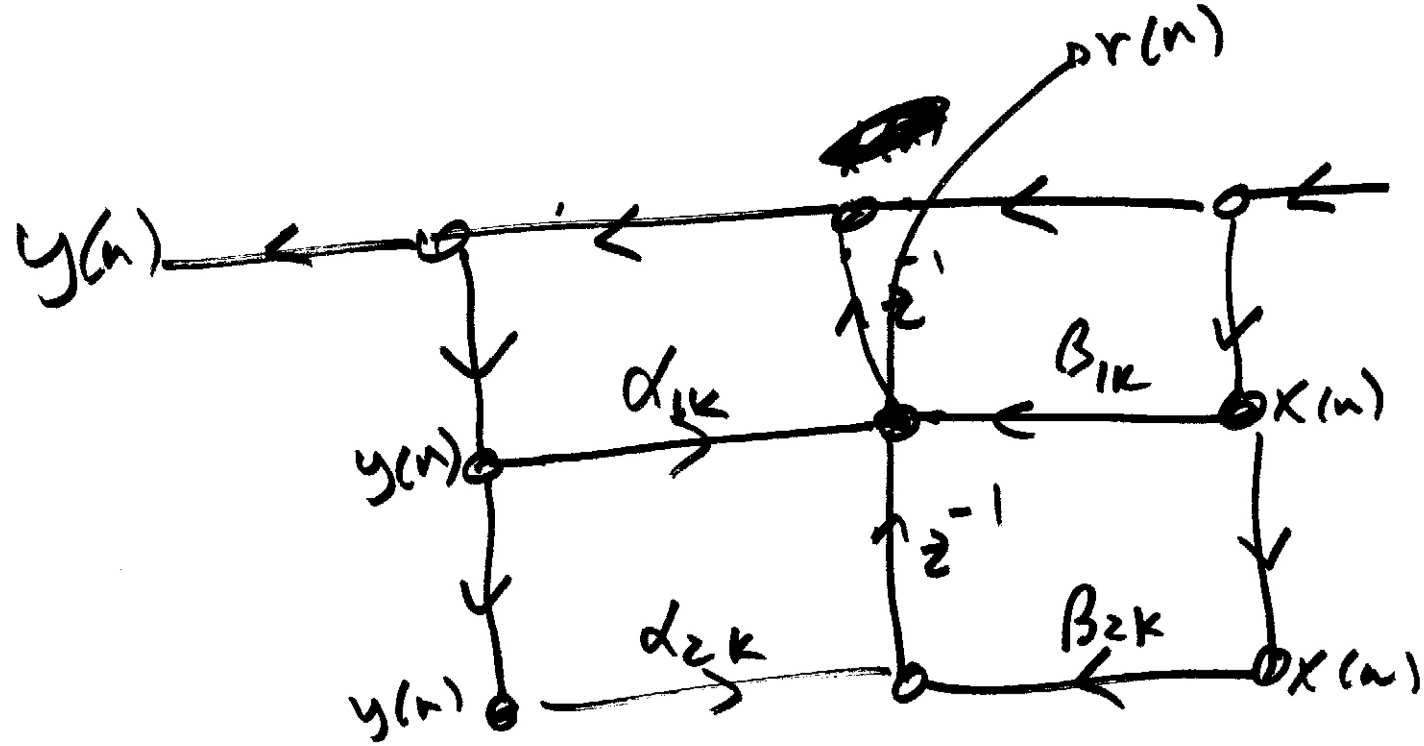
$$p(z_1, z_2) = \alpha z_1^2 + \beta z_2^2 + \gamma z_1 z_2 + \epsilon z_1 + \eta z_2 + \delta$$

Transposition Th.

change order of input/output ; change the direction of flow graph $\Longrightarrow$ get same system i.e. same input/output relationship.



$$H_k(z) = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 - \alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}}$$



$\frac{Y(z)}{X(z)}$

$$\begin{cases} r(n) = \alpha_{1k}\, y(n) + \beta_{1k}\, x(n) + \beta_{2k}\, x(n-1) \\ \qquad\quad + \alpha_{2k}\, y(n-1) \\ y(n) = x(n) + r(n-1) \end{cases}$$

$$y(n) = x(n) + \alpha_{1k}\, y(n-1) + \beta_{1k}\, x(n-1) + \beta_{2k}\, x(n-2) + \alpha_{2k}\, y(n-2)$$

$$\boxed{\frac{Y(z)}{X(z)} = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 - \alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}}} = H(z)$$

Realization of FIR filters

Filters

IIR

FIR.

Today.

X

IIR with Rational Transfer fn.

Direct form I

Direct form II

Cascade

Parallel.

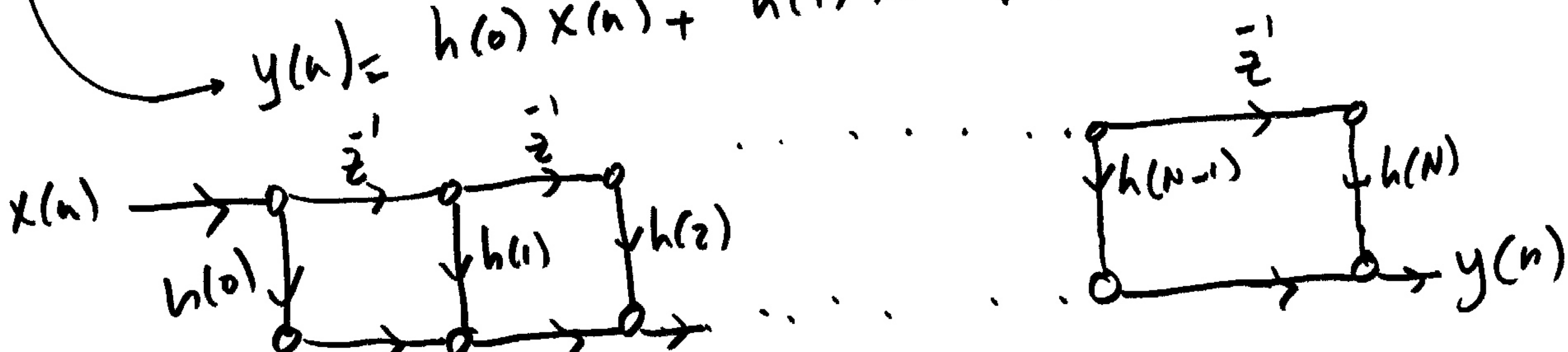
$$x \rightarrow \boxed{h(n)} \rightarrow y$$

$h(n)$ has $N+1$ taps.

$$y(n) = \sum_{k=0}^{N} h(k)\, x(n-k)$$

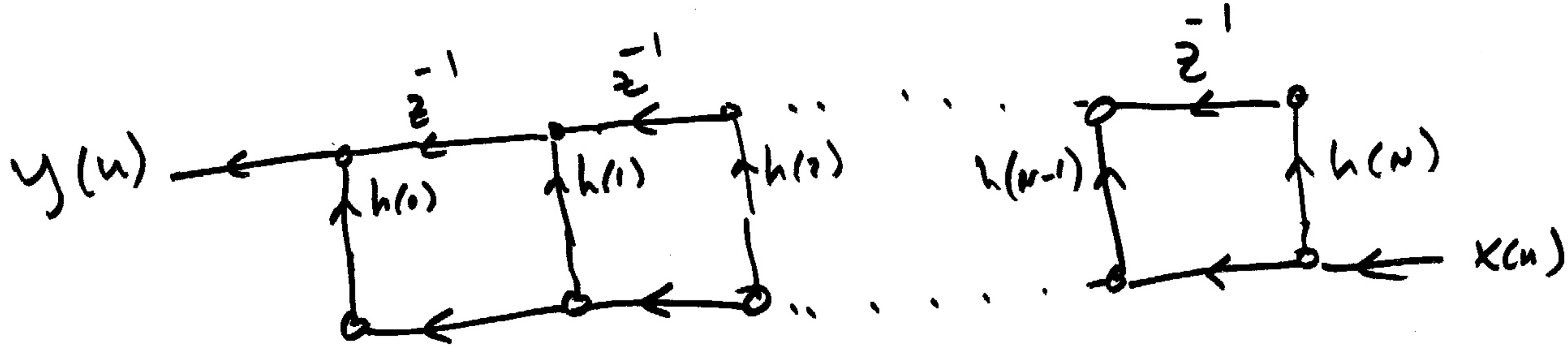
$$H(z) = \sum_{k=0}^{N} h(k)\, z^{-k}$$

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + \cdots + h(N)\,x(n-N)$$



Direct form I, II.

Transposed version of ⌋



Cascade.

$$H(z) = \sum_{k=0}^{N} h(k) z^{-k} = \prod_{k=1}^{N} \left(b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2} \right)$$