LECTURE 08

TRANSFORMS! Useful representations to signals and systems.

SO FAR:

Discrete to continuous: good for analysis

- DTFT → frequency analysis of signals/systems

- $z \rightarrow$ more general

Discrete to discrete: good for computing → practical

- DIF/DFS → periodic signals freq analysis.

- DFT → finite sequences
  
  fast computation
  
  fast convolutions
  
  practical freq analysis

- discrete short-time fourier transform → practical time-freq analysis.
**Discrete Transforms (Finite)**

DFT is only 1 out of a large class of transforms.

**Uses for: Analysis**
- Compression
- Denoising
- Detection
- Recognition

- Approximation (Sparse)

Sparse representation of signals has been one of the hottest research topics in the last 10 years. In SP

**Today:**

- Start with DFT → Frequency only
- Short-Time DFT → Time-Freq
- Wavelets → Better time freq.

Wavelets → Sparsity → Compression Denoising Approximation
Heisenberg Boxes

Time-frequency uncertainty principle

\[ T_\delta \cdot \Delta \omega \geq \frac{1}{2} \] for continuous time.

\[ \Delta \omega \cdot \Delta t = \Delta \frac{1}{T} \]

**DFT**

\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \]

\[ \Delta \omega = \frac{2\pi}{N} \]

\[ \Delta t = N \]

**Discrete STFT**

\[ X[n,k] = \sum_{m=0}^{2L-1} x[n+m] e^{-j\frac{2\pi km}{2L}} \]

\[ \Delta \omega = \frac{2\pi}{L} \]

\[ \Delta t = L \]
RECONSTRUCTION

\[ x[n+m] \omega_2[m] = \frac{1}{N} \sum_{k=0}^{N-1} X[n,k] e^{j \frac{2\pi km}{N}} \]

For non-overlapping windows \( R=L \)

\[ x[n] = \frac{x[n-RR]}{\omega[n-RR]} \]

what is the problem?

For stable reconstruction must overlap \( R \) window 50% at least

- works for ham, bartlett

Reconstruction with overlap and add. No division!
APPLICATION: NOISE REMOVAL

RECALL BERD CHIRPING

\( \rightarrow \) show audio waveform \( \rightarrow \) all occupied

\( \rightarrow \) can't separate noise signal

\( \rightarrow \) show spectrum \( \rightarrow \) can't separate either

\( \rightarrow \) show spectrogram \( \rightarrow \) sparse!

\[ \text{CAN IMPLEMENT ADAPTIVE FILTER} \]

\[ \text{ON \ldots JUST THRESHOLD!} \Rightarrow \text{NON-LINEAR} \]

LIMITATIONS OF DISCRETE FFT:

\( \rightarrow \) Need overlapping \( \Rightarrow \) Non-Orthogonal

\( \rightarrow \) Computationally intensive! For sequence

\[ \text{with: length} \ M \]
\[ \text{window} \ L \]
\[ \text{FFT} \ N \]
\[ \text{overlap} \ L \]

\( \rightarrow \) SAME SIZE Heisenberg boxes.
FROM STFT TO WAVELETS:

BASIC IDEA:
1. LOW FREQ, CHANGE SLOWLY - fast tracking not important
2. Fast tracking of high-freq important in many applications

Adopt Heisenberg box to freq.

BACK TO CONTINUOUS TIME for a second...

\[ Sf(u, \Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t} dt \]

translated window.

\[ \psi(u) = \int_{-\infty}^{\infty} f(t)\sqrt{\frac{1}{\psi}} \psi^*(\frac{t-u}{\psi}) dt \]

* Morlet* - G. Grassmann

\( \psi \) is called a mother wavelet

must satisfy:
1. \[ \int |\psi(t)|^2 dt = 1 < \infty \]
2. \[ \int \psi(t) dt = 0 \Rightarrow \text{Band-Pass} \]
Examples of Wavelets:

1) Mexican hat

\[ \psi(b) = (1 - t^2) e^{-\frac{t^2}{2}} \]

Murenzi (1989) used in computer vision.

2) Haar

\[ \psi(b) = \begin{cases} 
-1 & 0 \leq t < \frac{1}{2} \\
1 & \frac{1}{2} \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]

3) Show Mexican hat wavelet of u-p signal.
can be written as linear filtering:

\[ Wf(u,s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} \hat{f}(t) \psi^*_s \left( \frac{t-u}{s} \right) dt = \]

\[ = \hat{f}(t) \ast \psi^*_s (t) (u) \]

\[ \psi^*_s(t) = \frac{1}{\sqrt{s}} \psi^* \left( -\frac{t}{s} \right) \]

wavelet coefficients are a result of band pass filtering
(a) Many different constructions for different signals

- Haar good for piece-wise constant signals
- Battle-Levmarie' spline polynomials

(b) Can construct orthogonal wavelets.

\[ \sqrt{2} \psi \left( \frac{t - \frac{k}{2^n}}{2^j} \right) \text{ Haar is orth.} \]

(c) However, transform not useful for computation.

Let's go back to discrete.
BACK TO DISCRETE

1) EARLY 80's Theoretical work by Morlet, Grossman and Meyer mathematics and geophysics
2) Late 80's links to DSP by Daubechies and Mallat.

From CWT to DWT not so trivial
Must take care to maintain properties.

TIME FREQ AGAIN

Example: Haar-Wavelet

\[ W[k] = \sum_{n=0}^{N-1} x[n] \phi_k[n] \]
Haar for $N=3$

$\omega$

$\delta t$

$k=0$

$k=1$

approximation

detail

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Fast computation with filter banks.

\[ h[n] = \mu \]
\[ g[n] = \frac{1}{5} \]

Complexity is \( O(N) \) less than FFT!
Approximation from 25 coefficients

Haar

DFT
Denoising by thresholding
Noisy Wavelet Denoised