Multi-Dimensional Signal

- Our world is more complex than 1-D.
- Images: $f(x,y)$
- Videos: $f(x,y,t)$
- Dynamic 3D scene $f(x,y,z,t)$
  - Medical imaging
  - 3D video
  - Computer graphics.

- we will focus on 2D

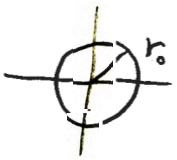
Continuous time functions

- $\delta(x,y)$ impulse function at $x=0, y=0$
- $\delta(x)$ impulse line (Vertical or Horizontal?)
- $\Pi(x,y) = \begin{cases} 1 & |x,y| < \frac{1}{2} \\ 0 & \end{cases}$

- $\cos(2\pi(f_x x + f_y y))$ - spatial harmonic

Circularly symmetric:

$\delta(r) \Rightarrow \delta(r-r_0)$ impulse ring.



$\Pi(r) \Rightarrow$

2D Fourier transform

what is a spatial frequency?

complex harmonic:

$$e^{j(\Omega_x x + \Omega_y y)} = e^{j(2\pi(f_x x + f_y y))}$$

$$\boxed{\text{show Examples of } \cos(2\pi(f_x x + f_y y))}$$

CT 2DFT:

$$F(f_x, f_y) = \iint f(x,y)\, e^{-j2\pi(f_x x + f_y y)}\, dx\, dy.$$

DTFT

$$F(\omega_x, \omega_y) = \sum_{n_x} \sum_{n_y} f[n_x, n_y]\, e^{-j(\omega_x n_x + \omega_y n_y)}$$

$$F(k_x, k_y) = \sum_{n_x} \sum_{n_y} f[n_x, n_y]\, e^{-j2\pi(n_x k_x + n_y k_y)}$$

$$= \underbrace{\sum_{n_x} \underbrace{\left(\sum_{n_y} f[n_x, n_y]\, e^{-j2\pi n_y k_y} \right)}_{\text{1D-DTFT in } y} e^{-j2\pi n_x k_x}}_{\text{1D-DTFT in } x} =$$

$$= \sum_{n_y} \left(\sum_{n_x} f[n_x, n_y]\, e^{-j2\pi n_x k_x} \right) e^{-j2\pi n_y k_y}$$

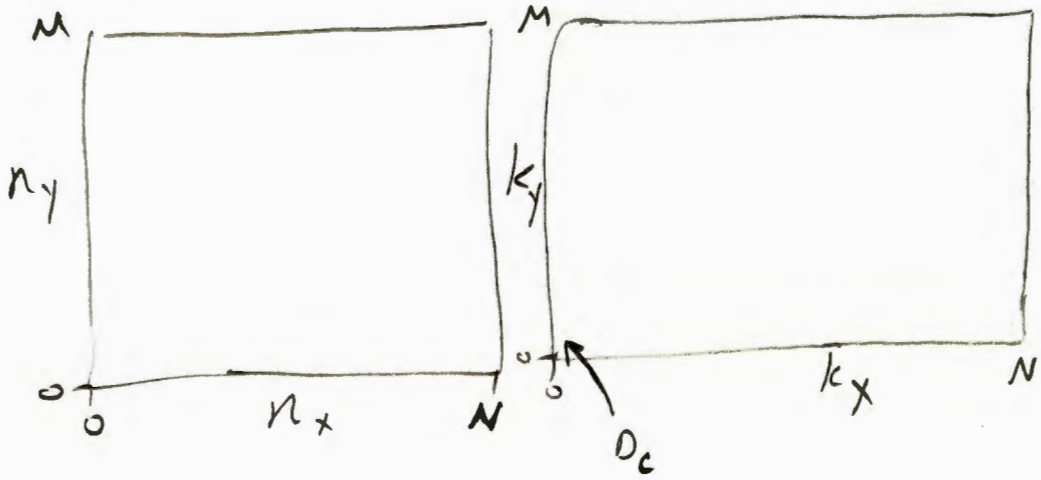
2D - DFT:

$$F[\tilde{k}_x, \tilde{k}_y] = \sum_{n_x=0}^{N-1} \sum_{n_y=0}^{M-1} f[n_x, n_y]\, e^{-j2\pi\left(\frac{n_x \tilde{k}_x}{N} + \frac{n_y \tilde{k}_y}{M}\right)}$$

$$\boxed{k_x = \frac{\tilde{k}_x}{N}}$$

2D - IDFT:

$$f[n_x, n_y] = \frac{1}{NM} \sum_{k_x=0}^{N-1} \sum_{k_y=0}^{M-1} F[\tilde{k}_x, \tilde{k}_y]\, e^{j2\pi\left(\frac{n_x \tilde{k}_x}{N} + \frac{n_y \tilde{k}_y}{M}\right)}$$



Properties: SIMILAR TO 1D

circular convolution:

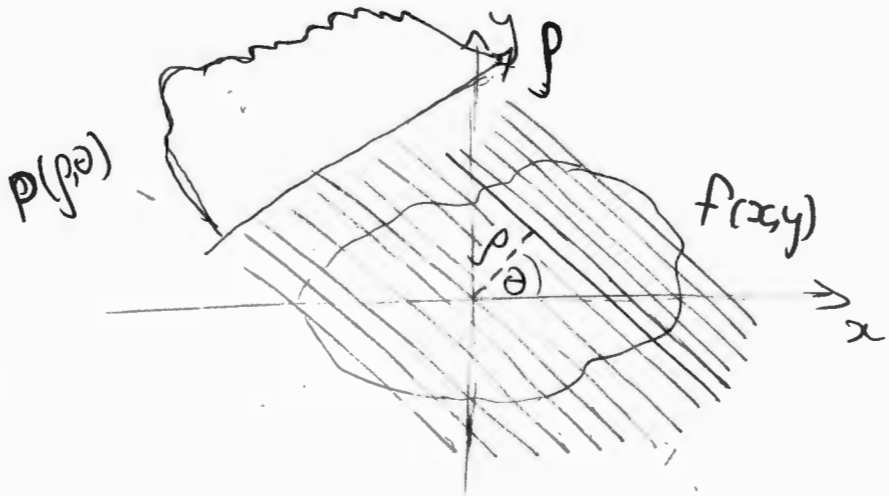
$$f[n_x, n_y] \circledast h[n_x, n_y] = F[\tilde{k}_x, \tilde{k}_y]\, H[\tilde{k}_x, \tilde{k}_y]$$

$$\Updownarrow$$

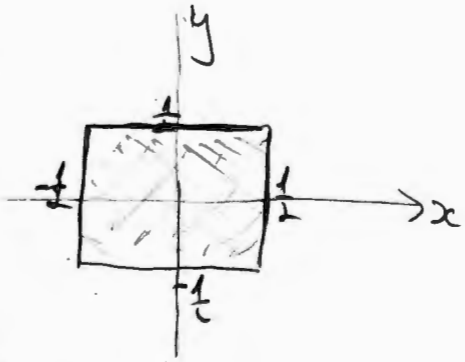
$$f[(n_x - m_x)_N, (n_y - m_y)_M] = e^{-j2\pi\left(\frac{\tilde{k}_x m_x}{N} + \frac{\tilde{k}_y m_y}{M}\right)} F[\tilde{k}_x, \tilde{k}_y]$$

PROJECTIONS : Radon Transform.

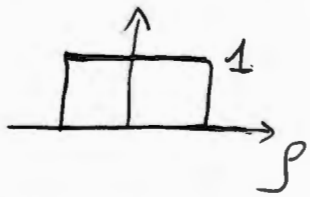
$$p(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(\rho - x\cos\theta - y\sin\theta)\,dx\,dy$$



FOR Example:

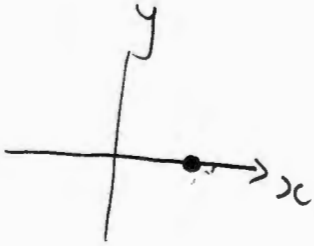


$p(\rho,0)$

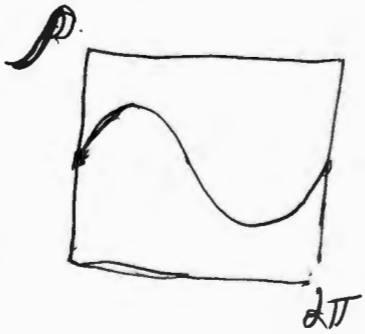


$p(\rho,\frac{\pi}{4})$

also called sinogram:



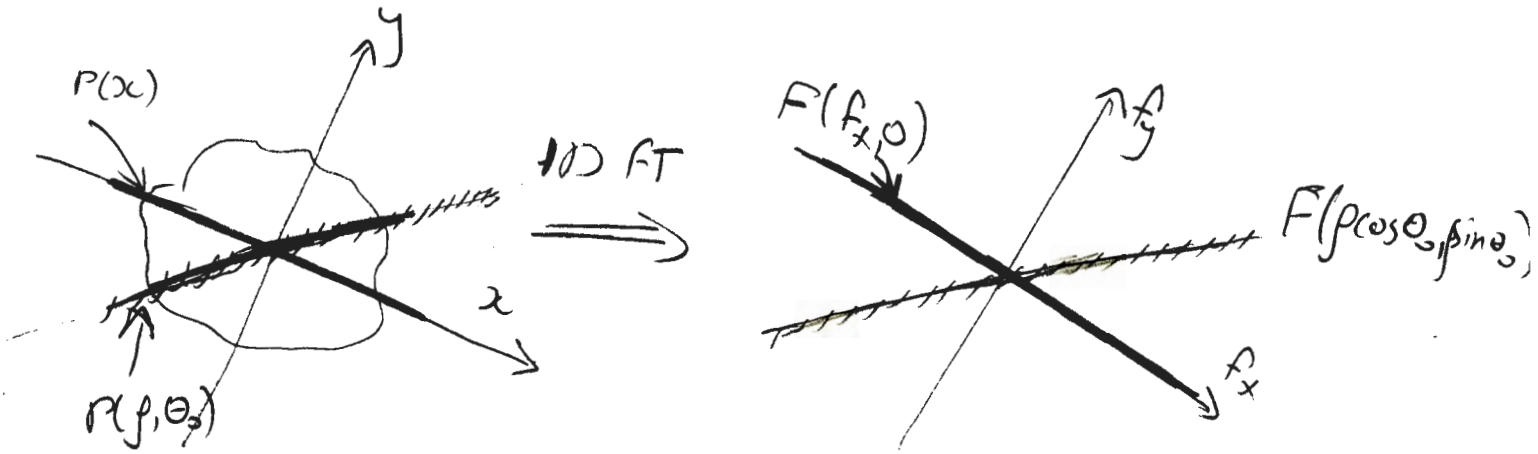
$p(\rho,\theta)$



Projection Slice Theorem R. Bracewell

$$\mathcal{F}_{1D}\{p(\rho,\theta_0)\} = F(\rho\cos\theta_0, \rho\sin\theta_0)$$

2D Fourier transform.



Proof: show only for $\theta = 0 \Rightarrow$ generalize to arbitrary θ

$$p(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$F(f_x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(f_x x + f_y y)}\,dx\,dy =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi f_x x}\,dx\,dy =$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right] e^{-j2\pi f_x x}\,dx =$$

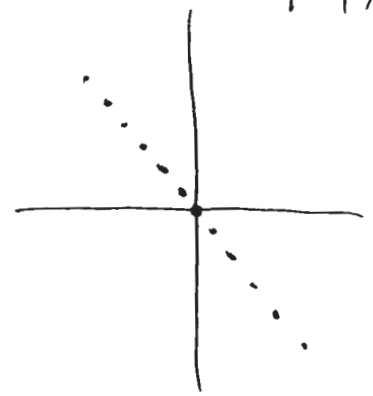
$$= \int_{-\infty}^{\infty} p(x)\, e^{-j2\pi f_x x}\,dx = \mathcal{F}\{p(x)\}$$

DISCRETE RECONSTRUCTION

$$P_{\theta_m}[\rho_n] \xrightarrow{\text{1D DFT}} P_{\theta_m}[\rho_k]$$

$\rho_n \to 0-[N-1]$, $\rho_k \to [0-N-1]$

2D DTFT $F(k_x, k_y)$



$$k_x = \frac{\omega_x}{2\pi} = \frac{\rho_k}{N}\cos\theta_m$$

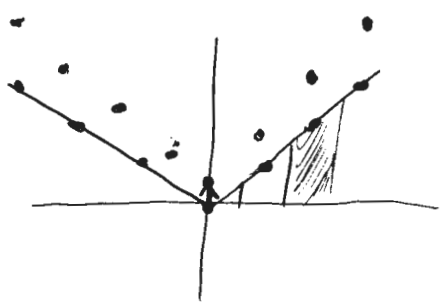
$$k_y = \frac{\omega_y}{2\pi} = \frac{\rho_k}{N}\sin\theta_m$$

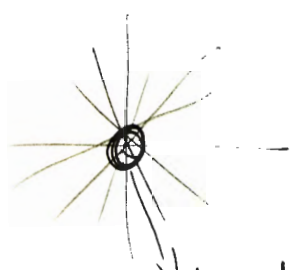
$$f[n,m] = \iint F(k_x,k_y)\, e^{-2\pi j(k_x n + k_y m)}\, dk_x\, dk_y =$$

$$= \iint F(\rho,\theta)\, e^{-2\pi j(\rho\cos\theta n + \rho\sin\theta m)}\, |\rho|\, d\rho\, d\theta$$

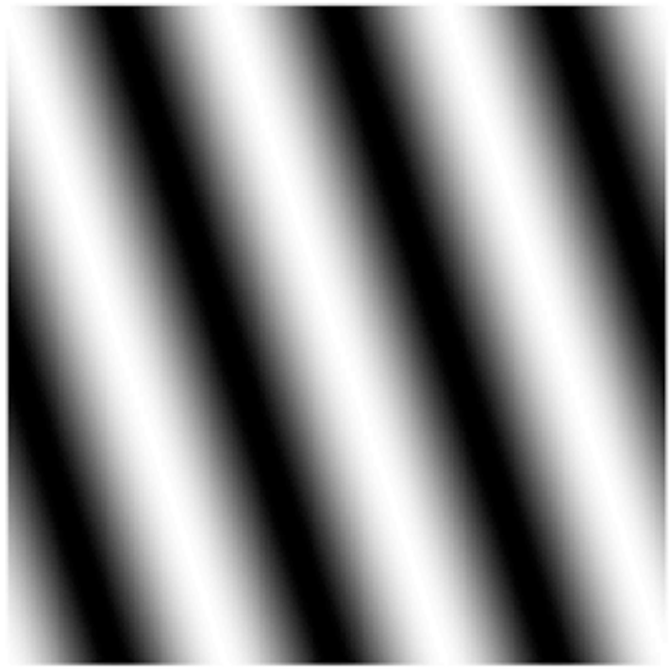
DISCRETIZE:

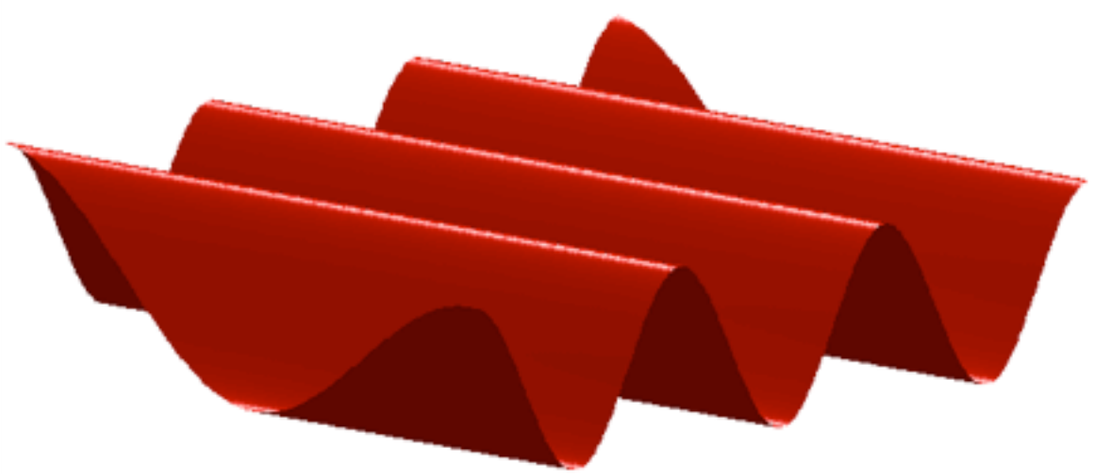
$$\simeq \sum\sum F\left(\frac{\rho_k}{N},\theta_m\right) e^{-2\pi j\left(\frac{\rho_k}{N}\cos\theta_n n + \frac{\rho_k}{N}\sin\theta_n m\right)} \left|\frac{\rho_k}{N}\right|$$

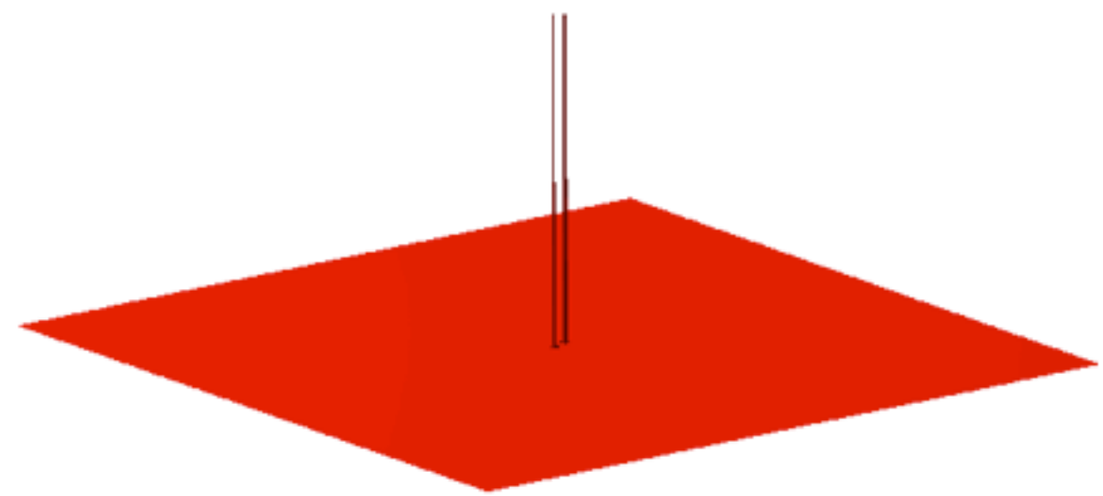


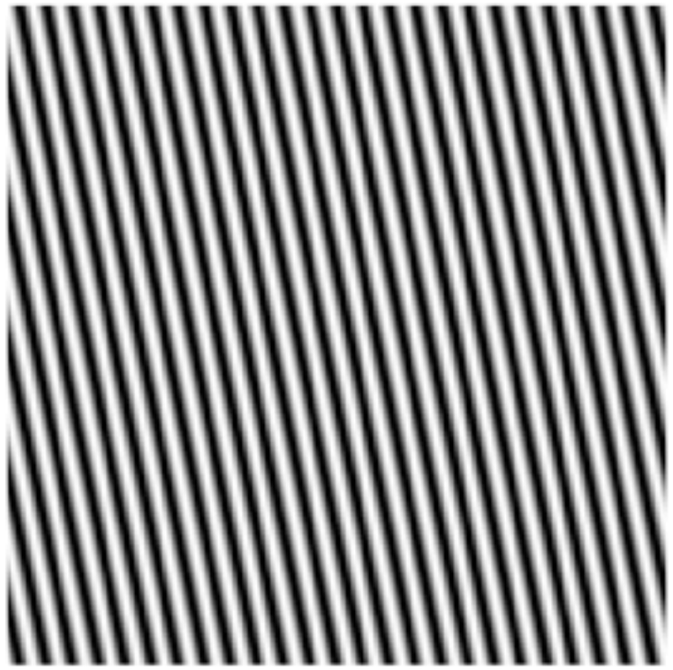


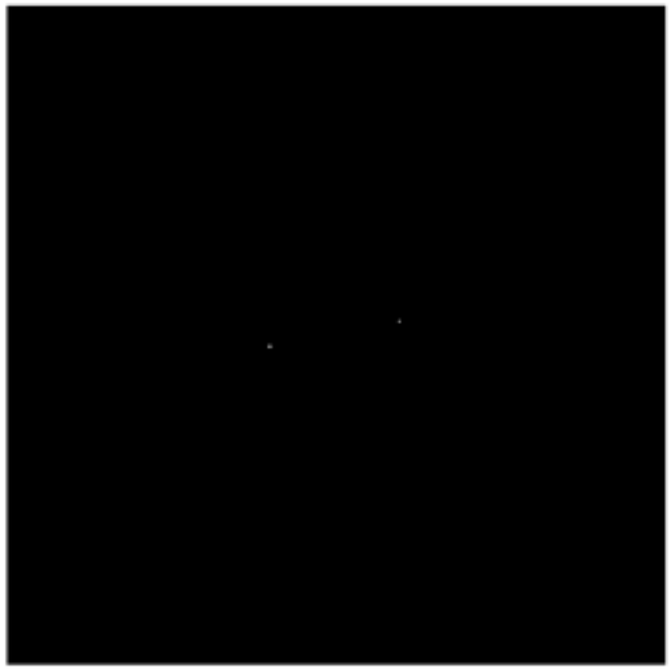
high dens.

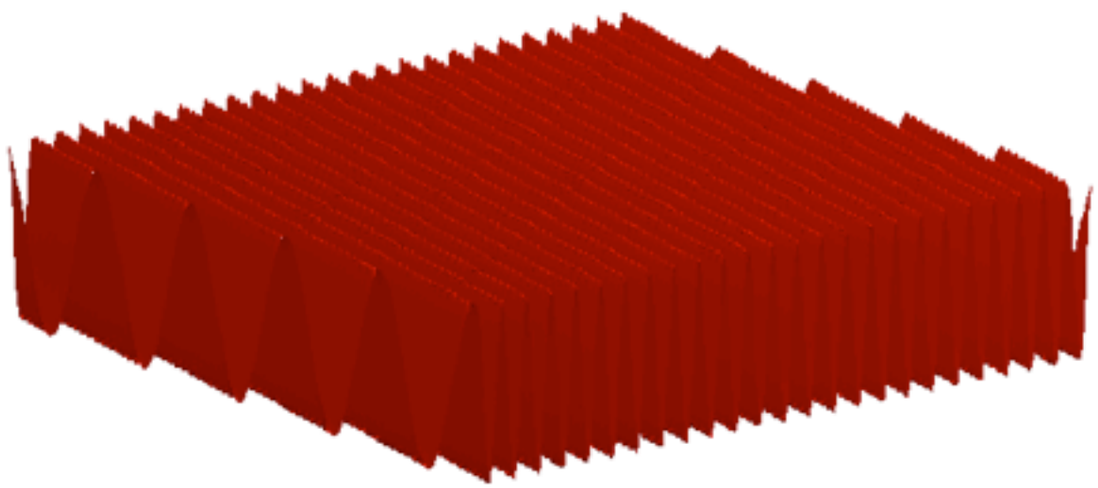


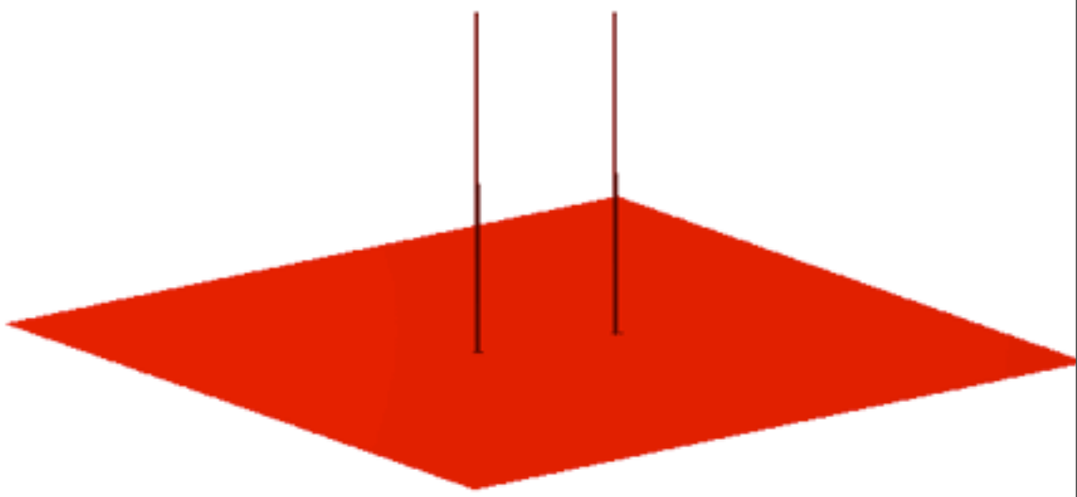


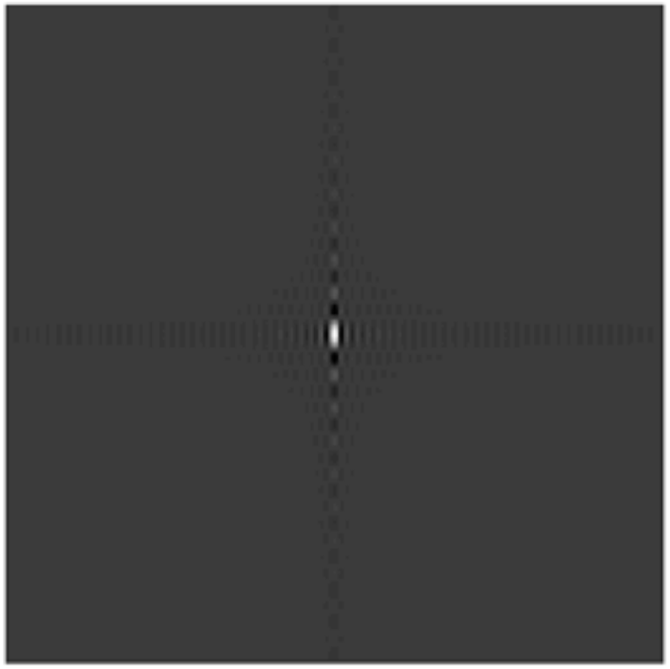


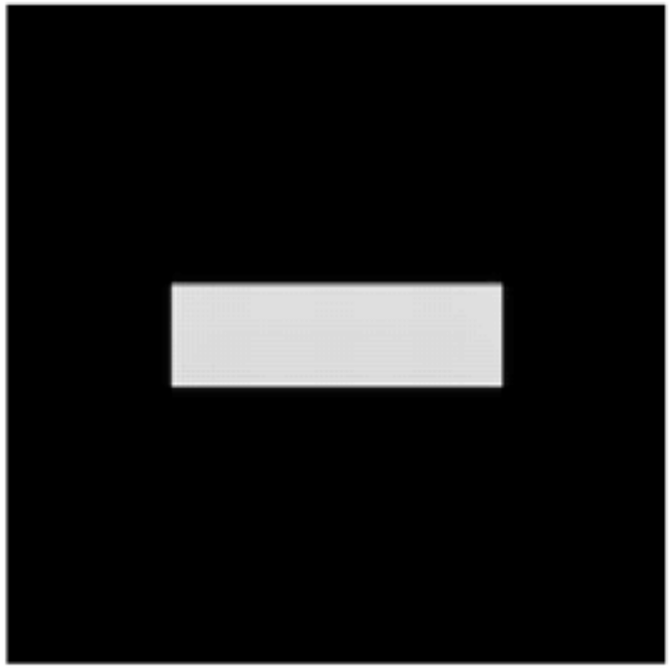


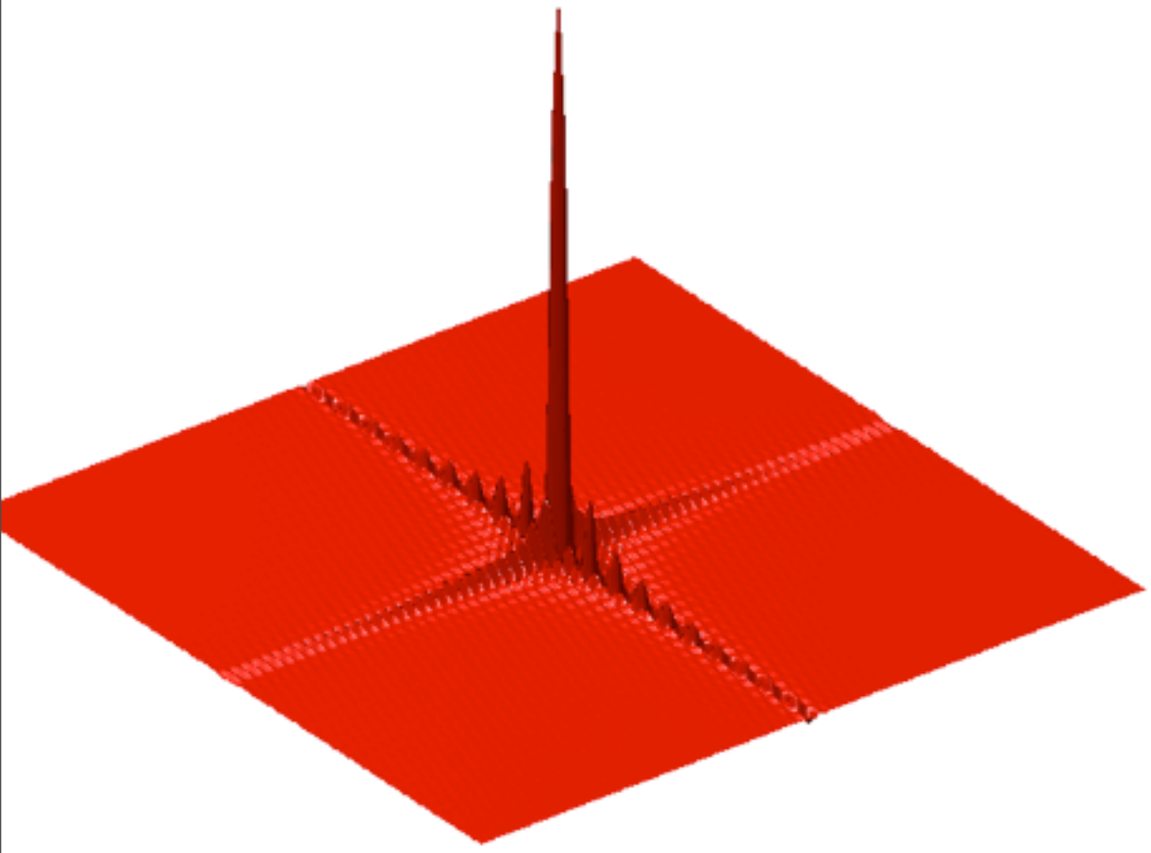


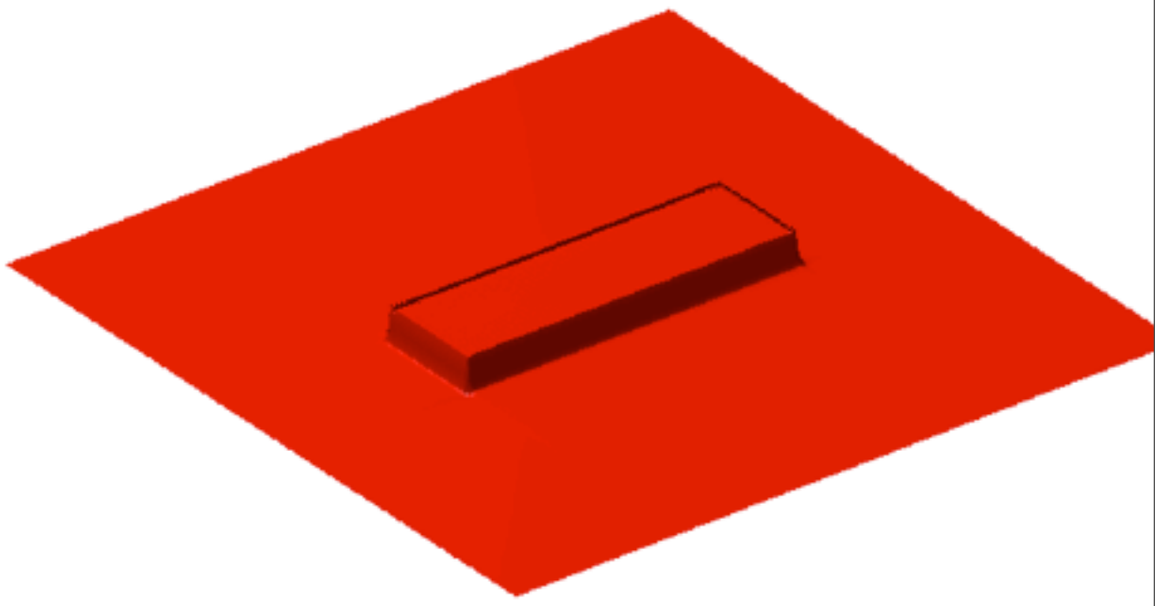


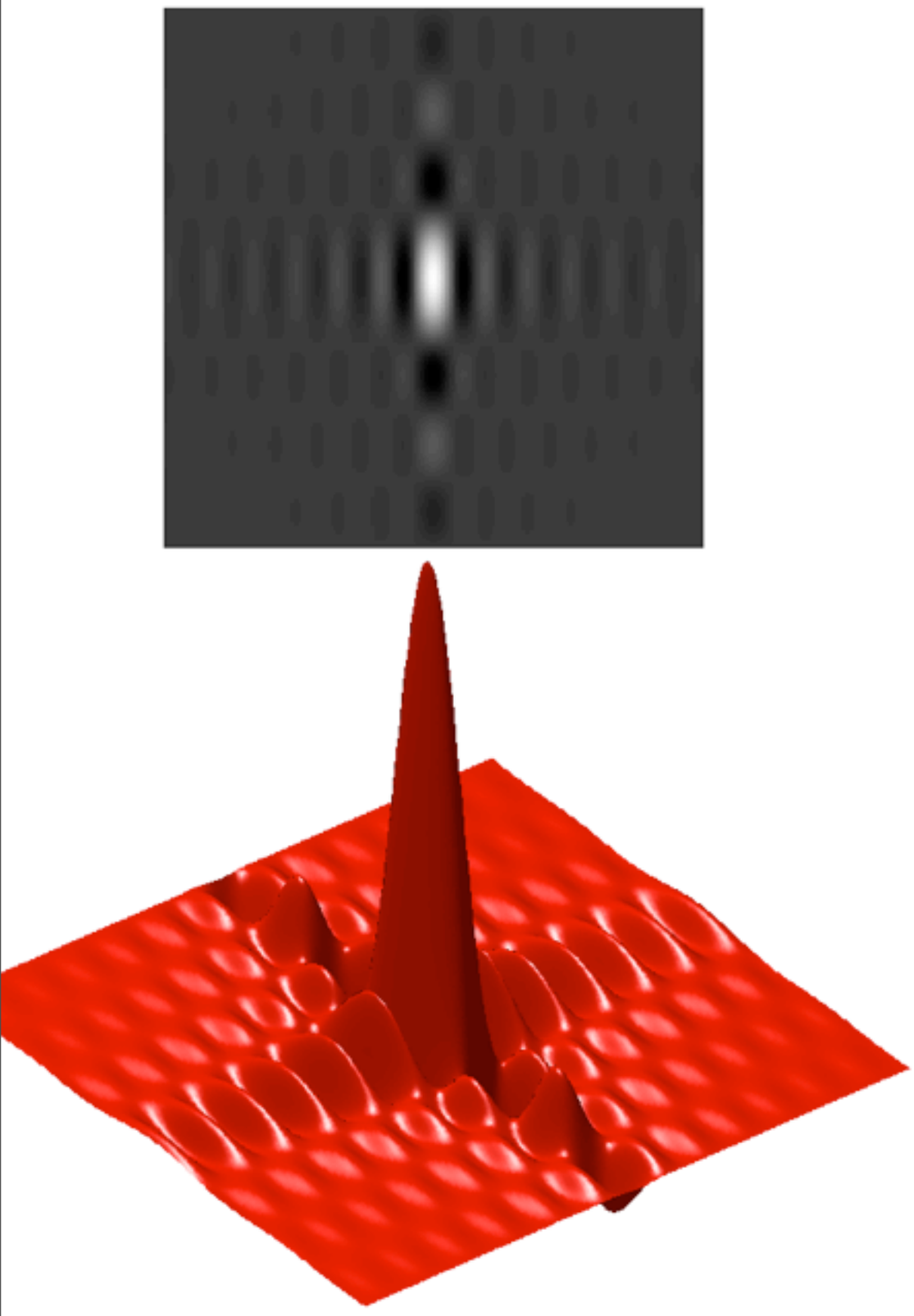


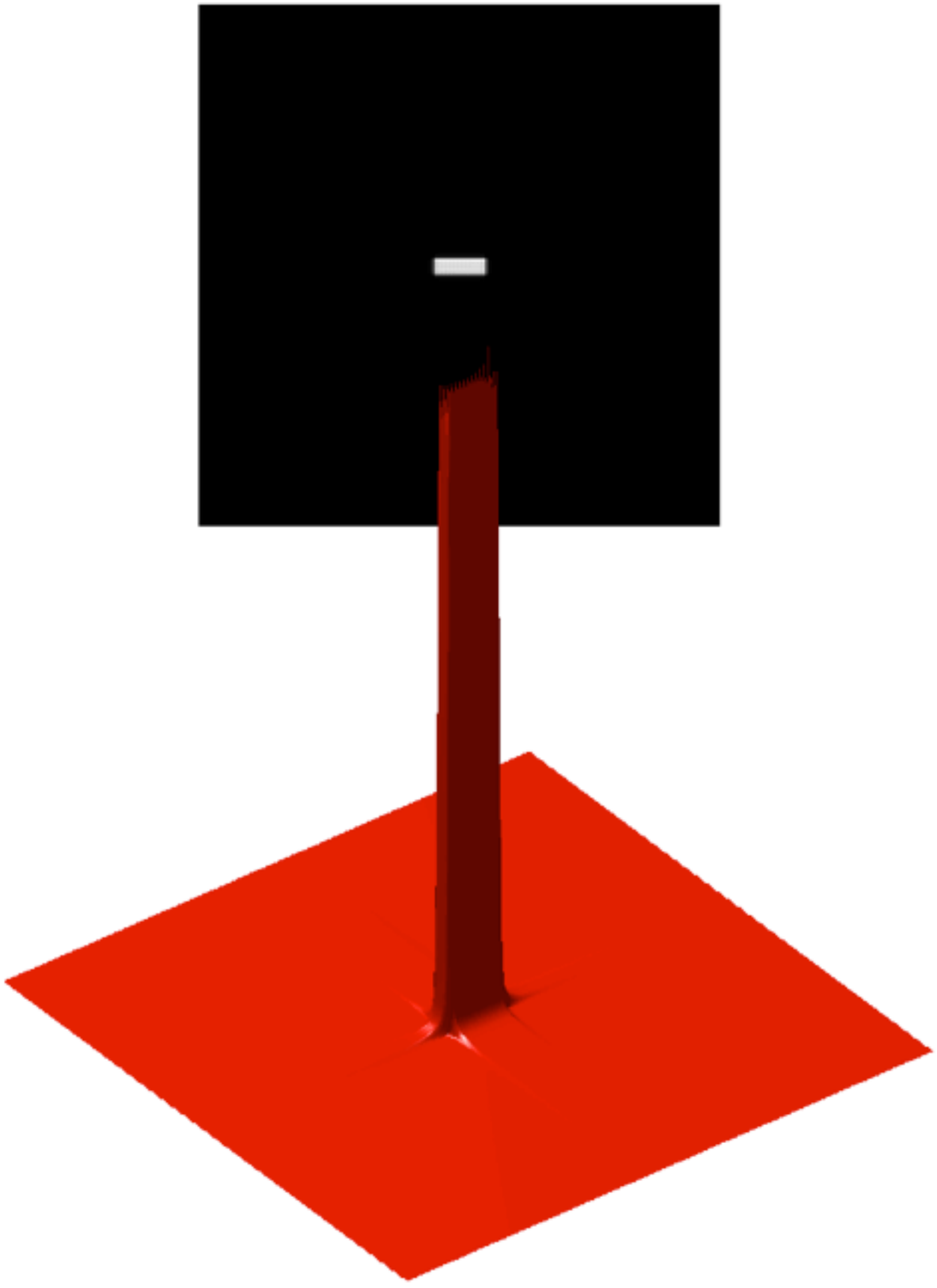




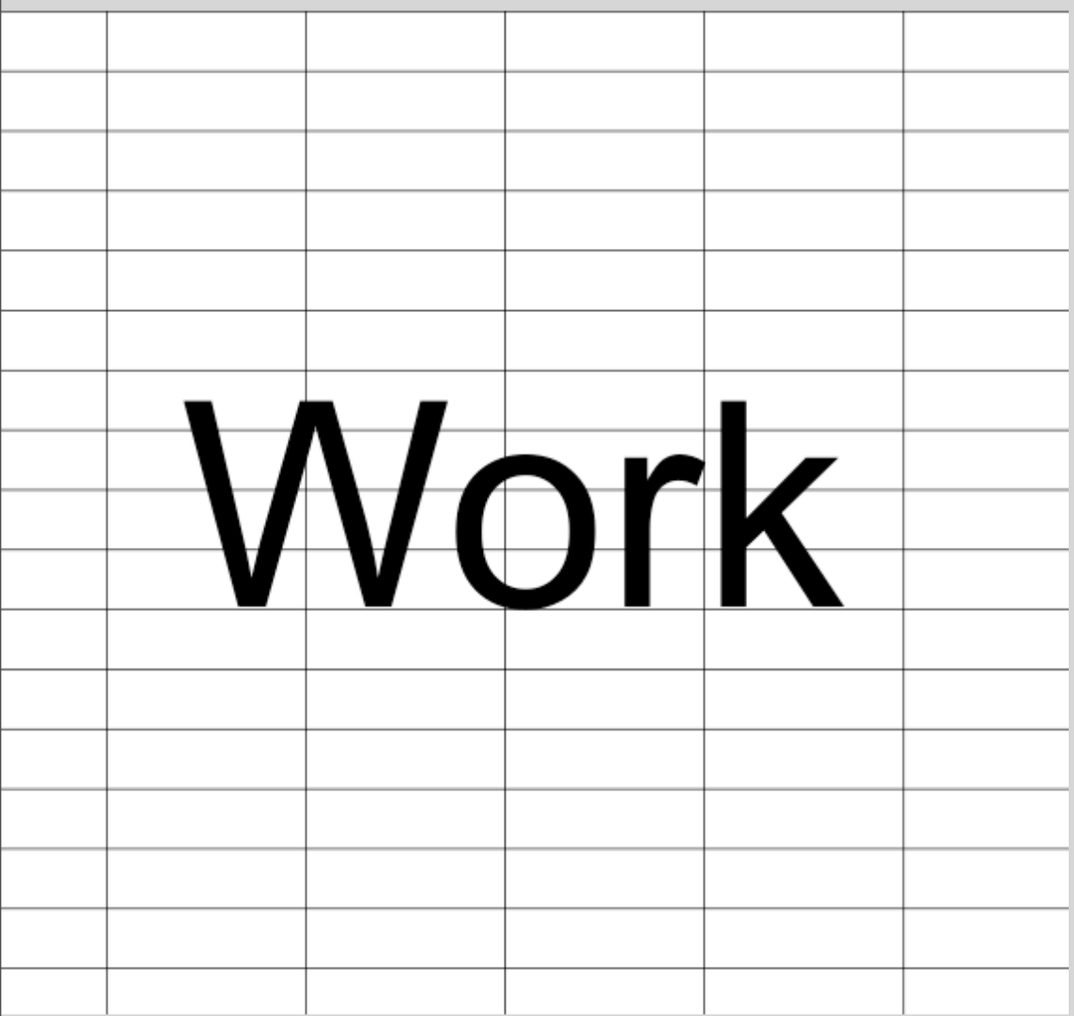




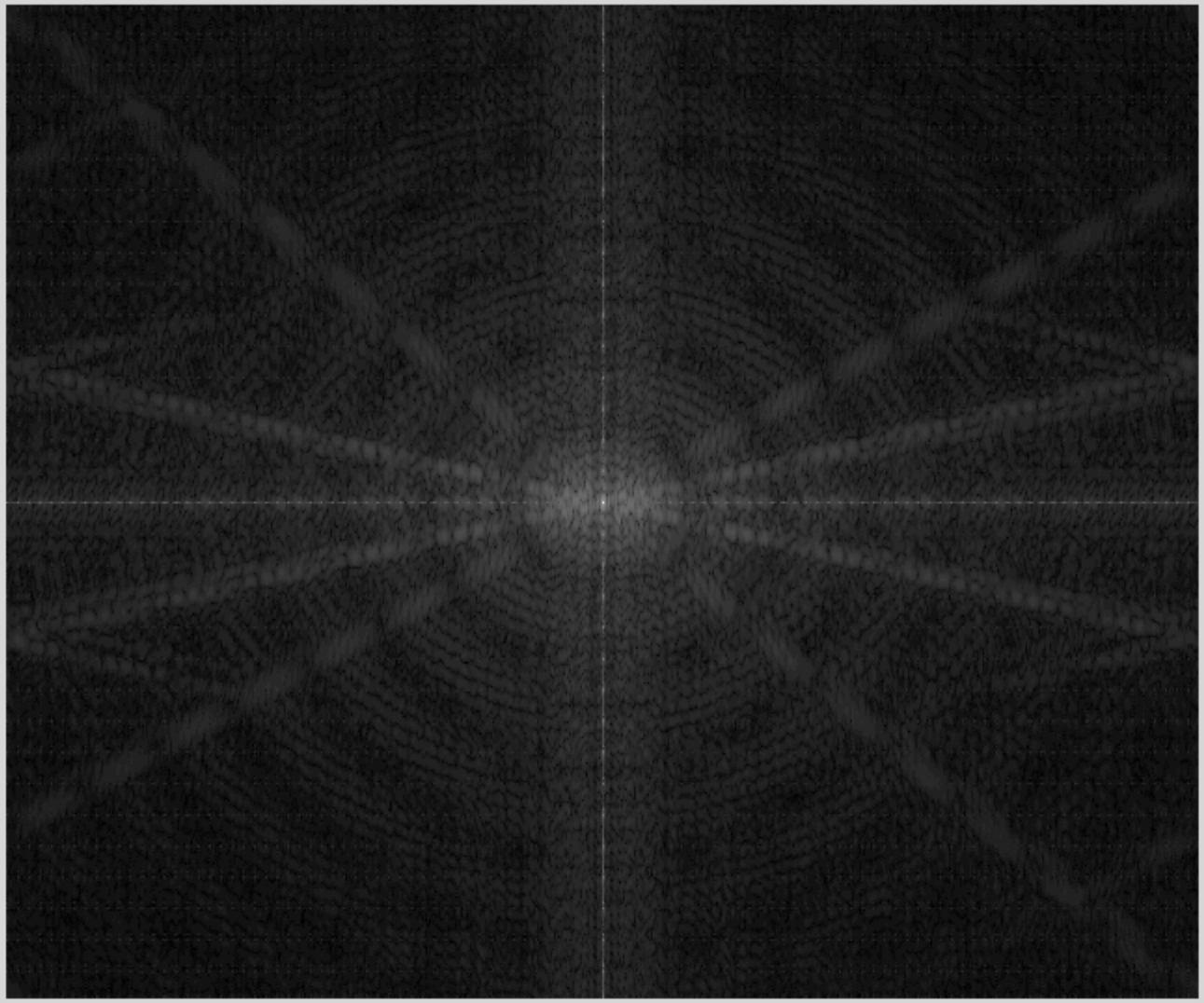




f[x,y]



|F[kx,ky]|



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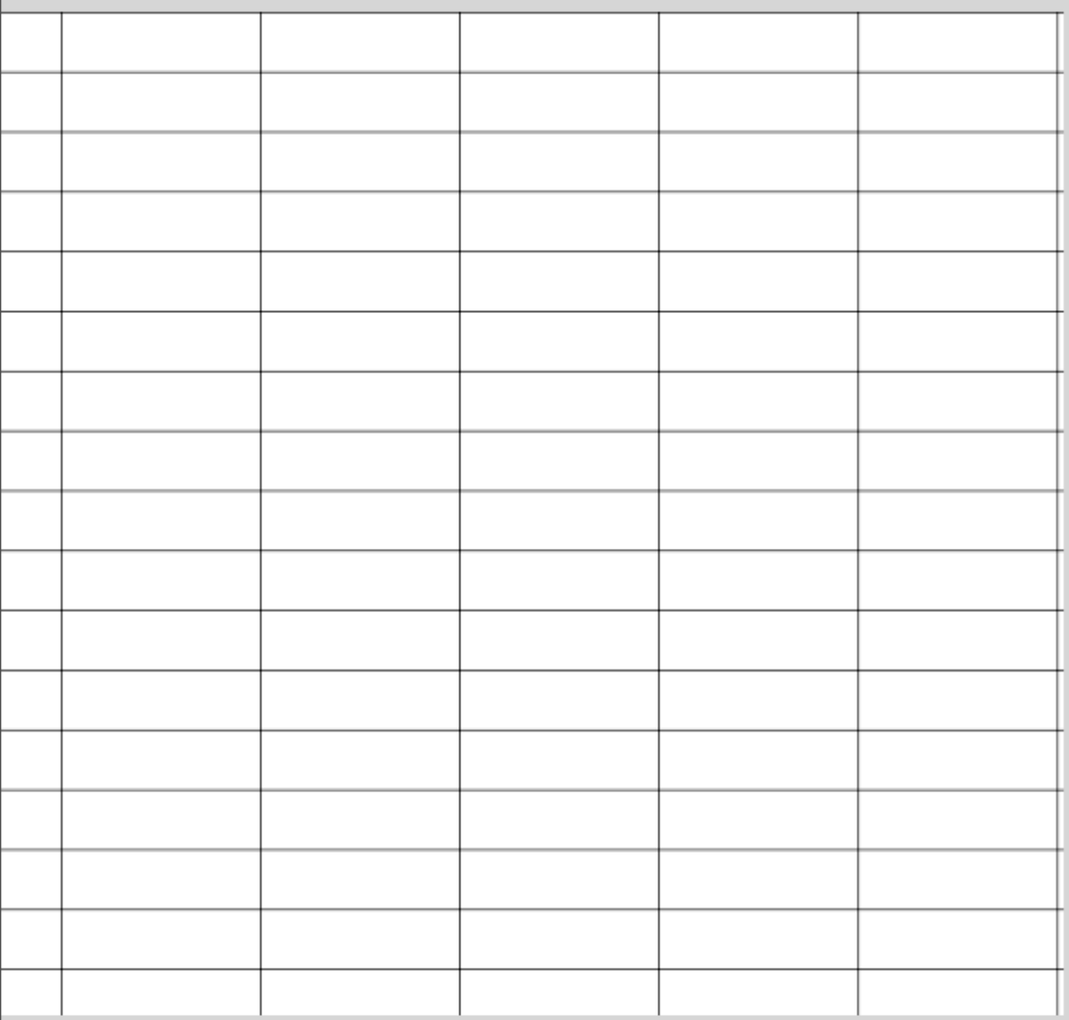

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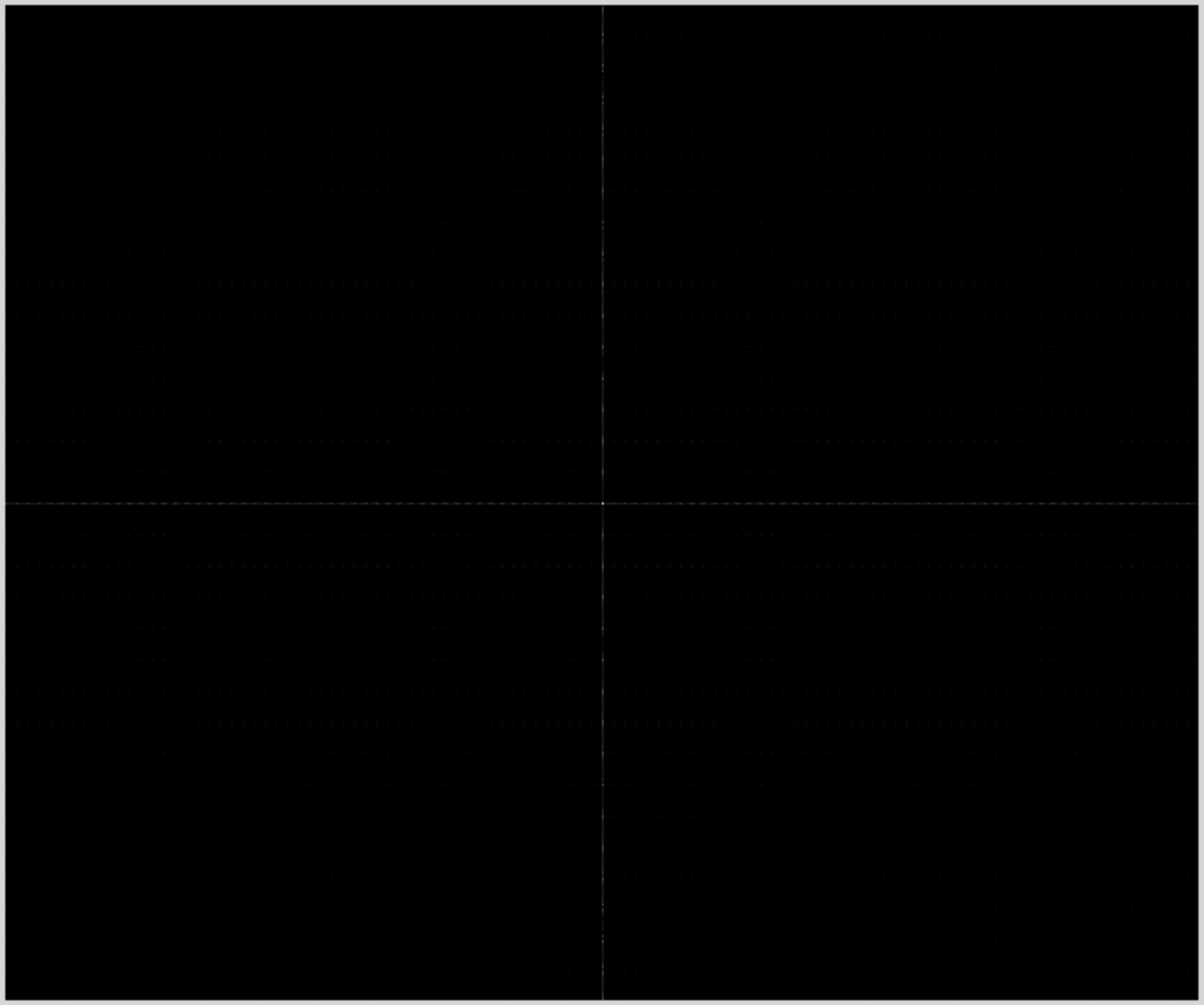
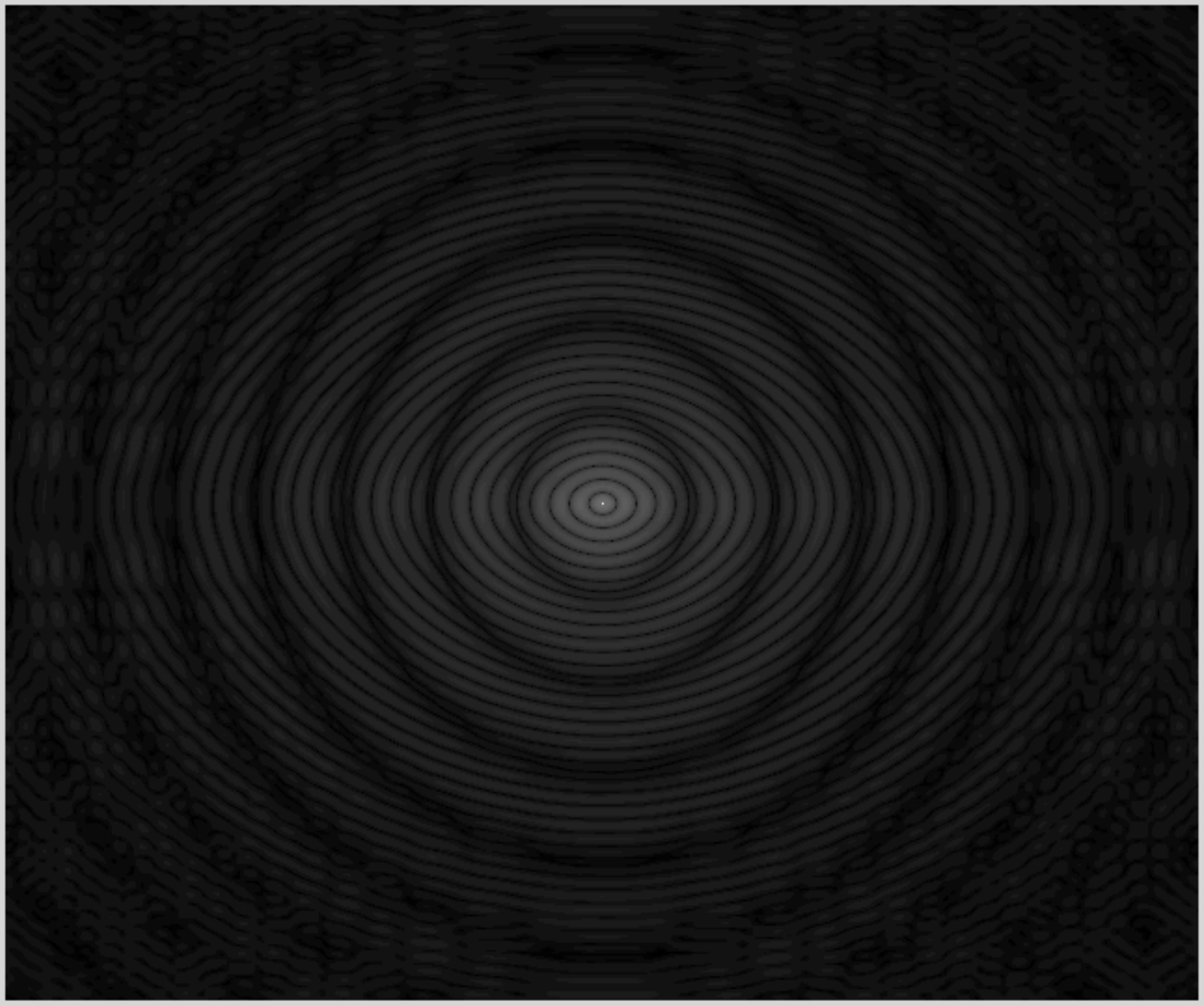


f[x,y]

O

|F[kx,ky]|



f[x,y]

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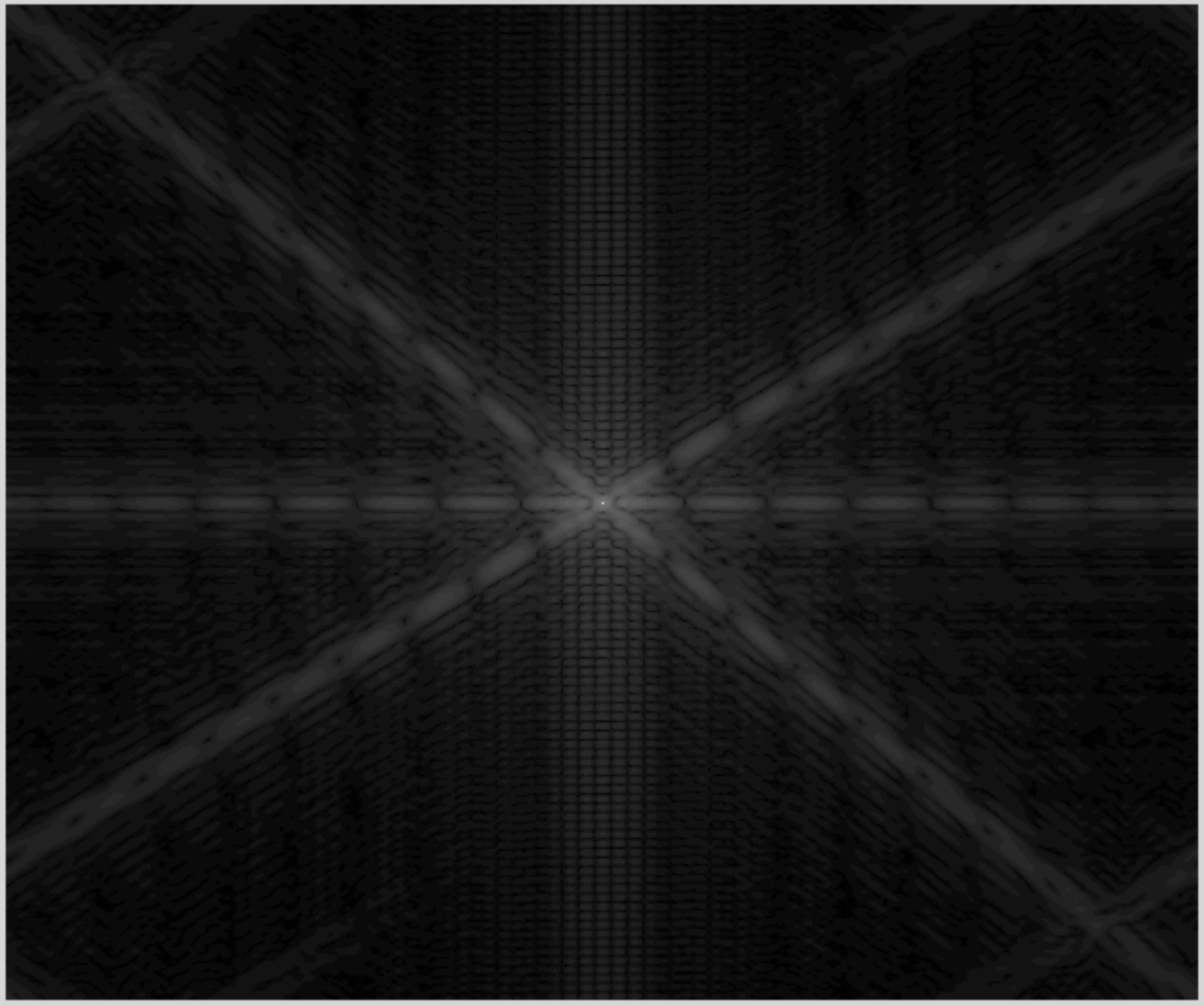
|F[kx,ky]|



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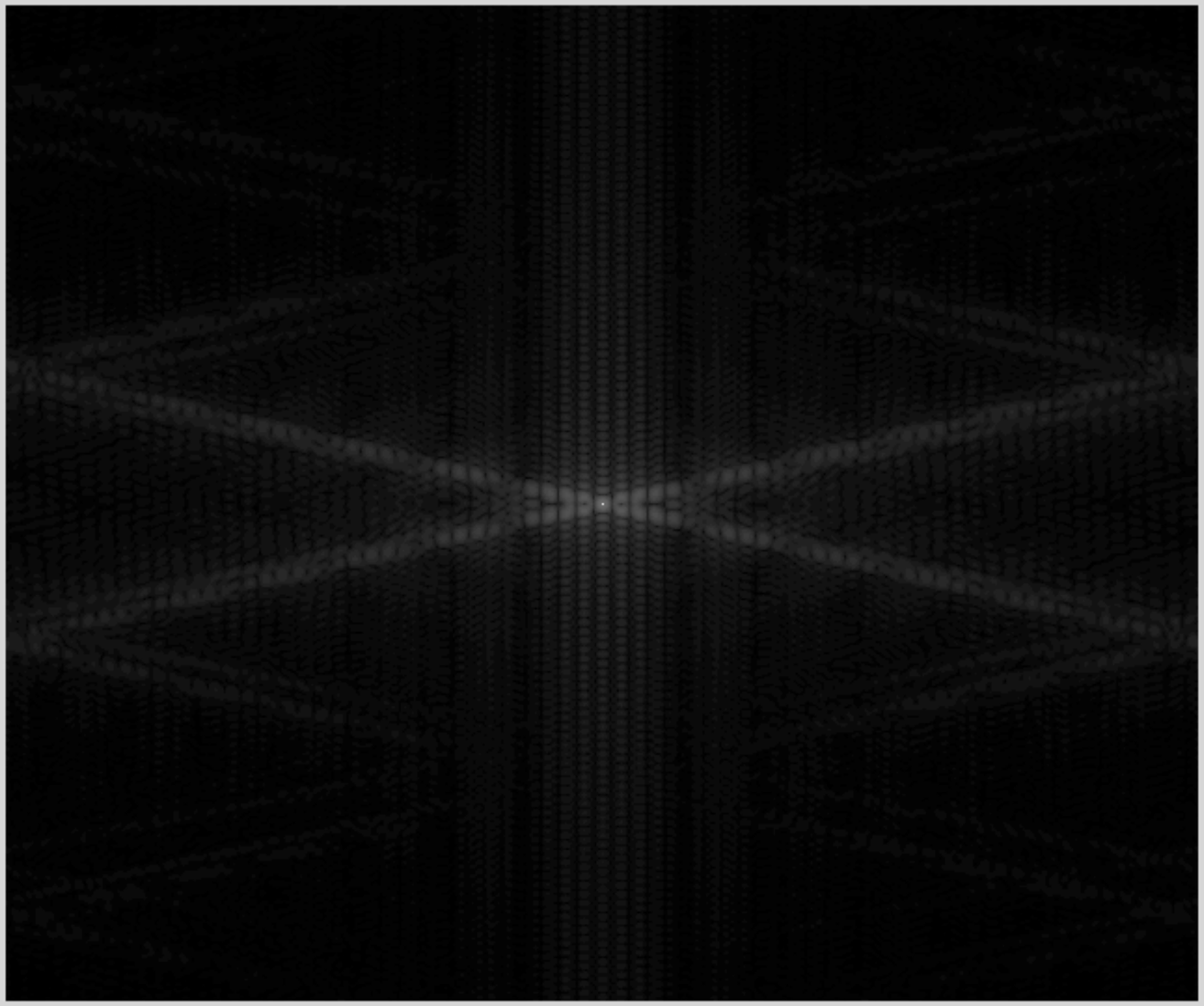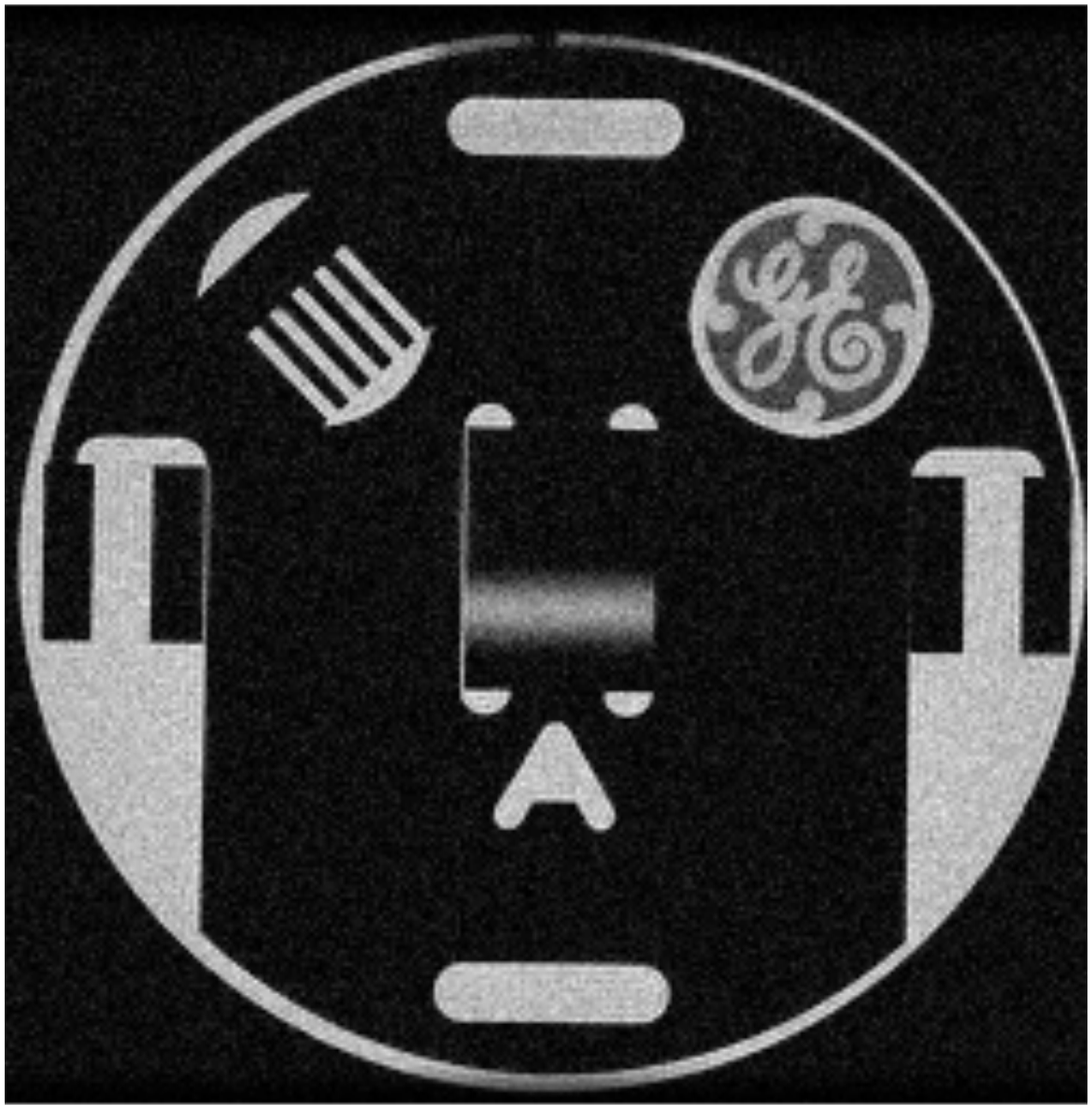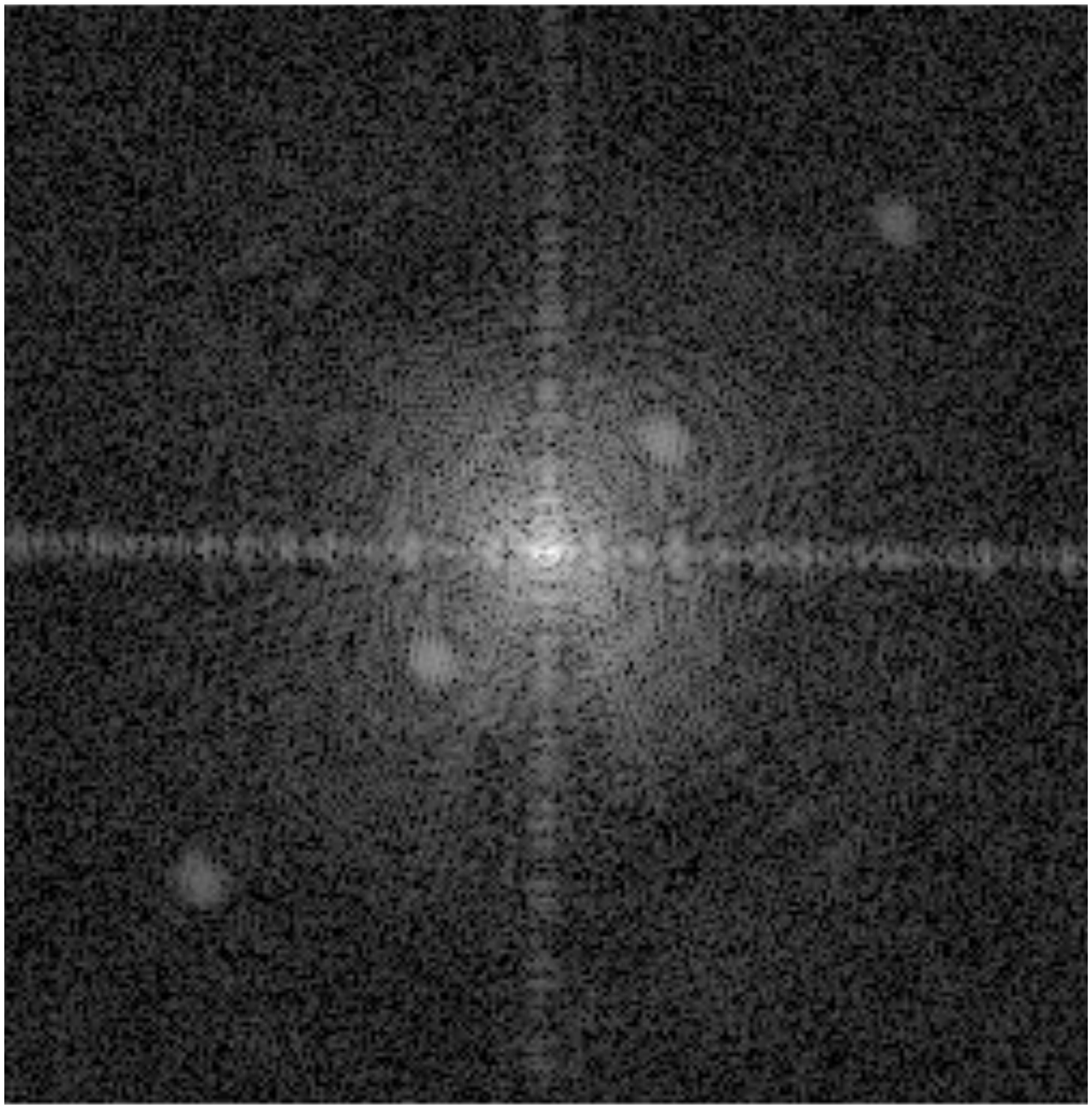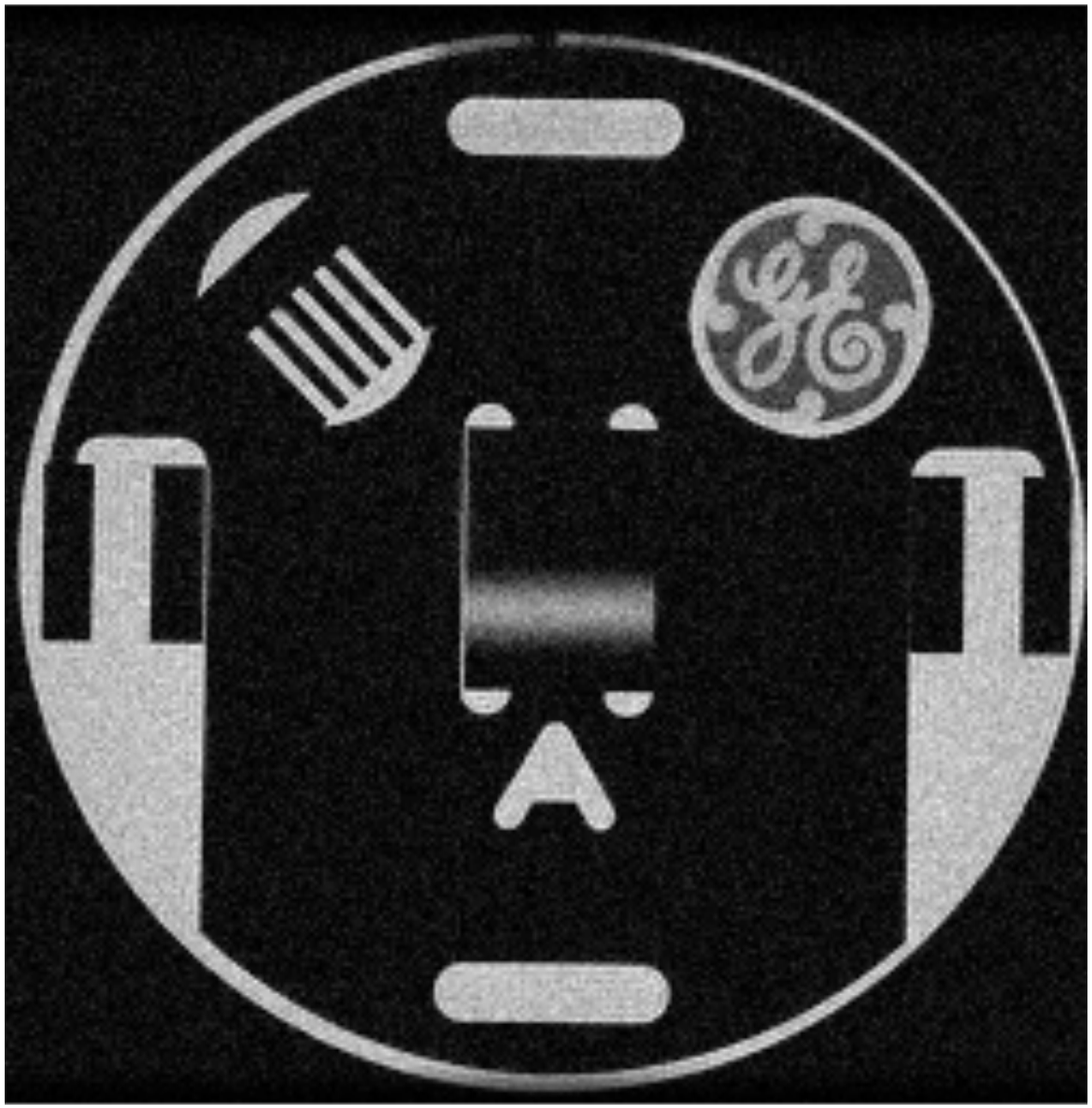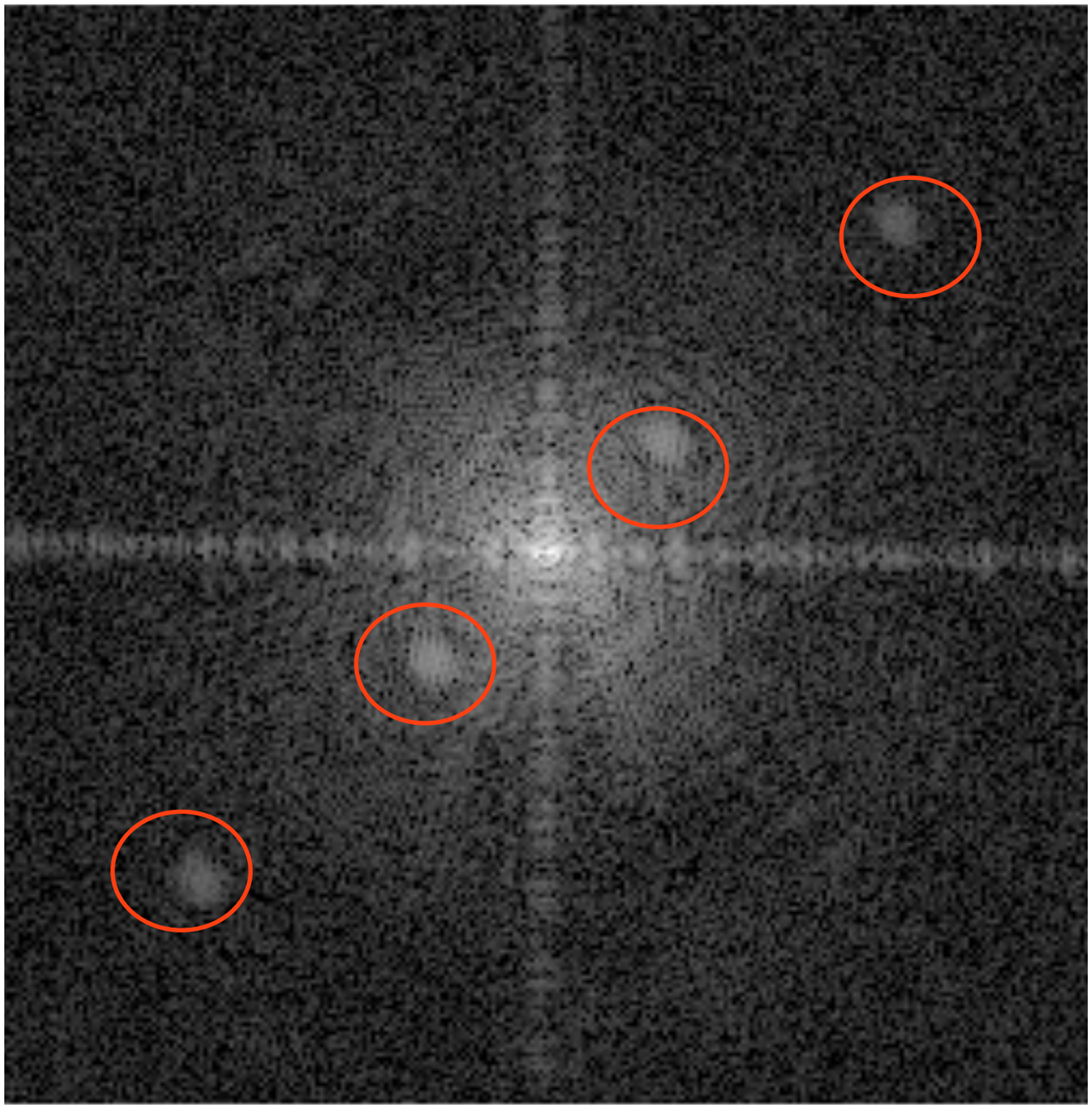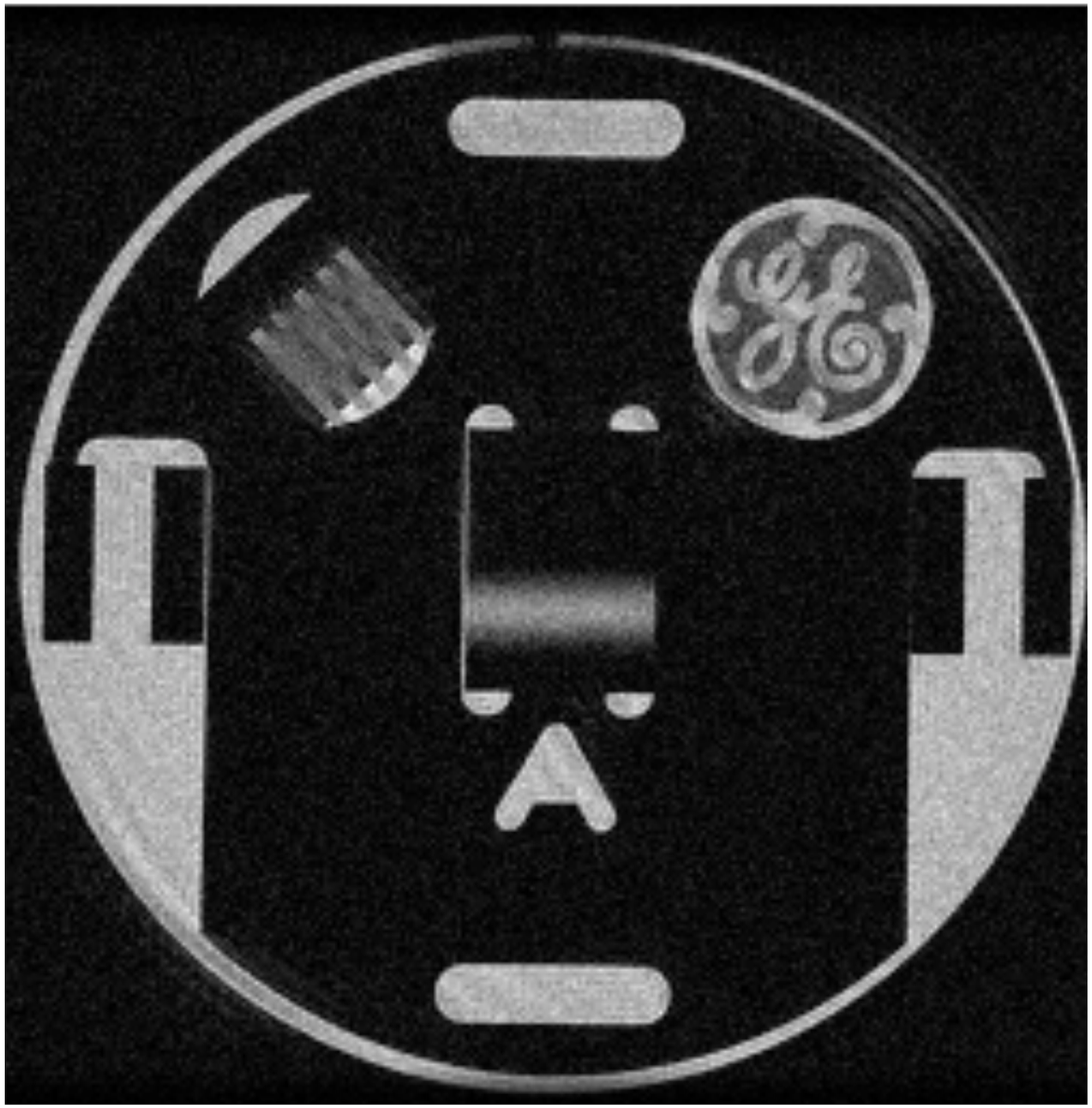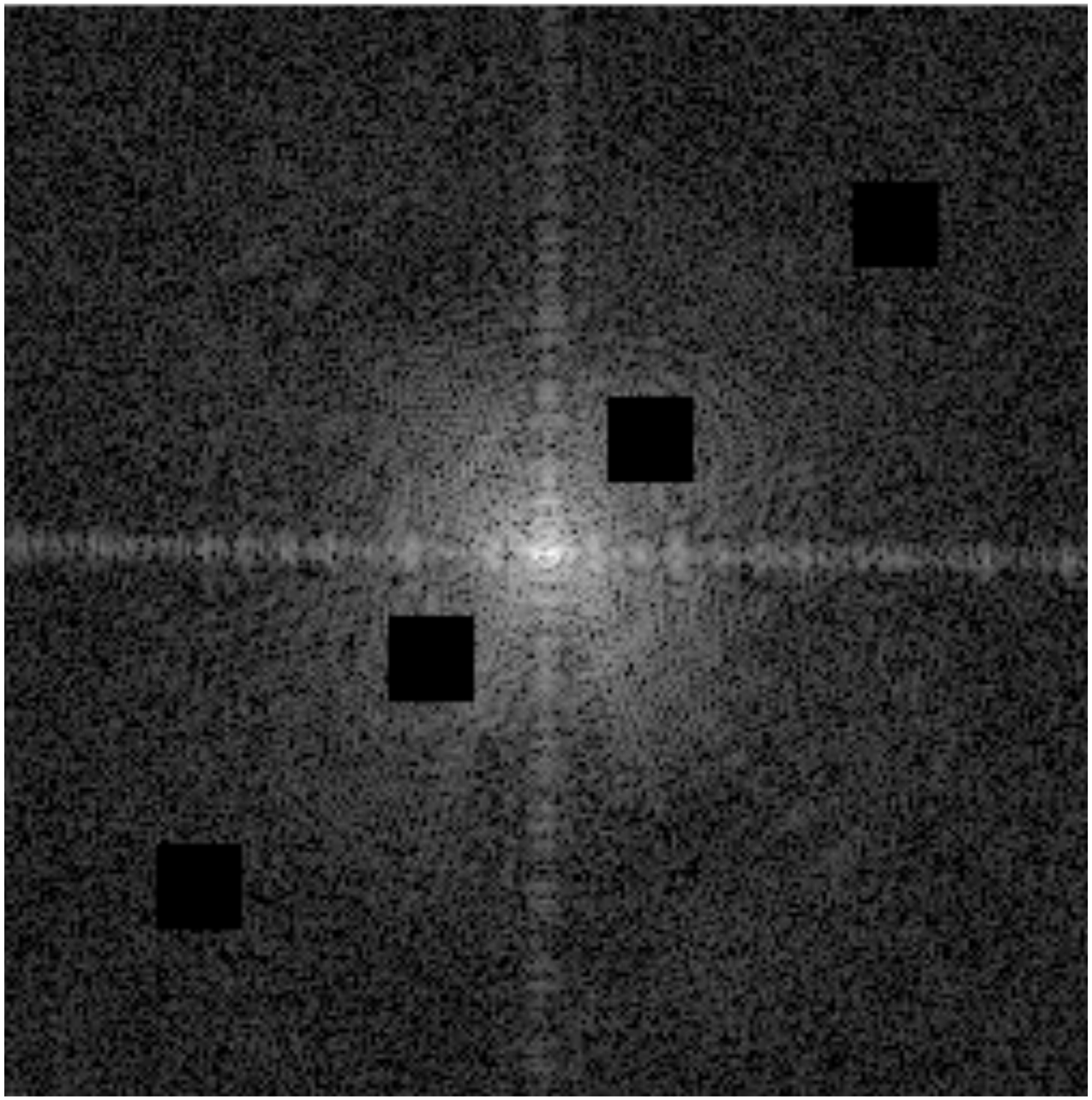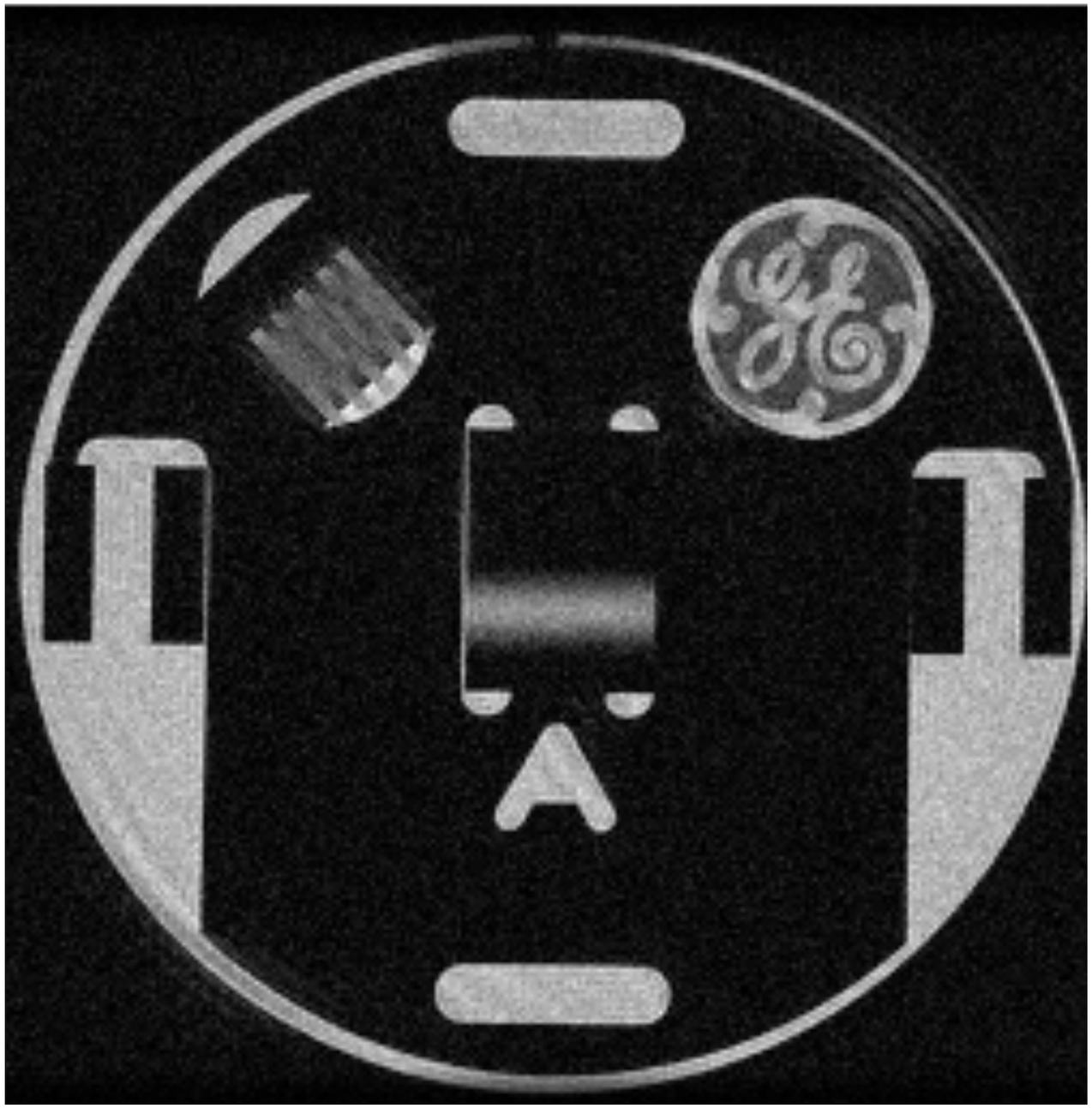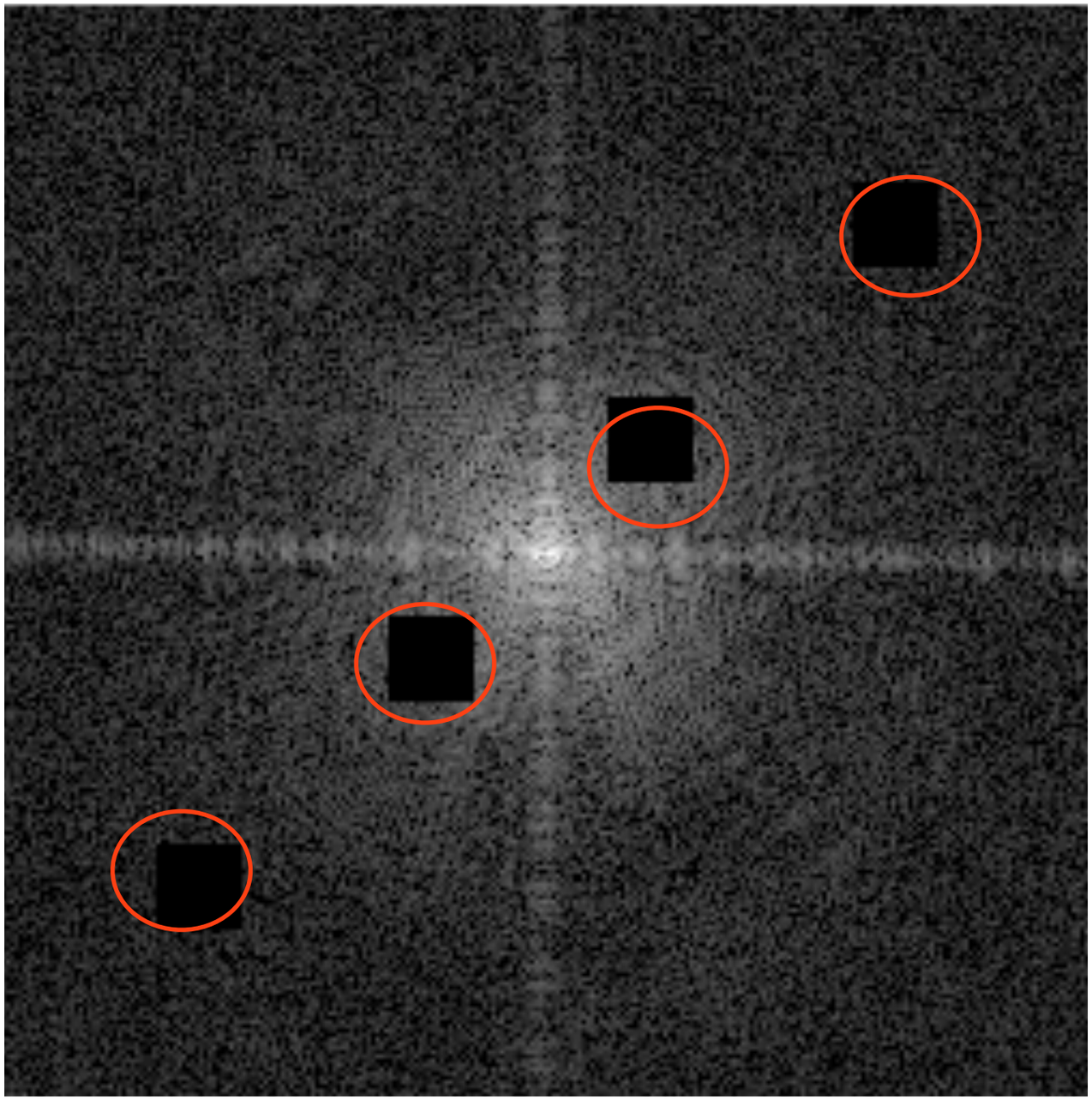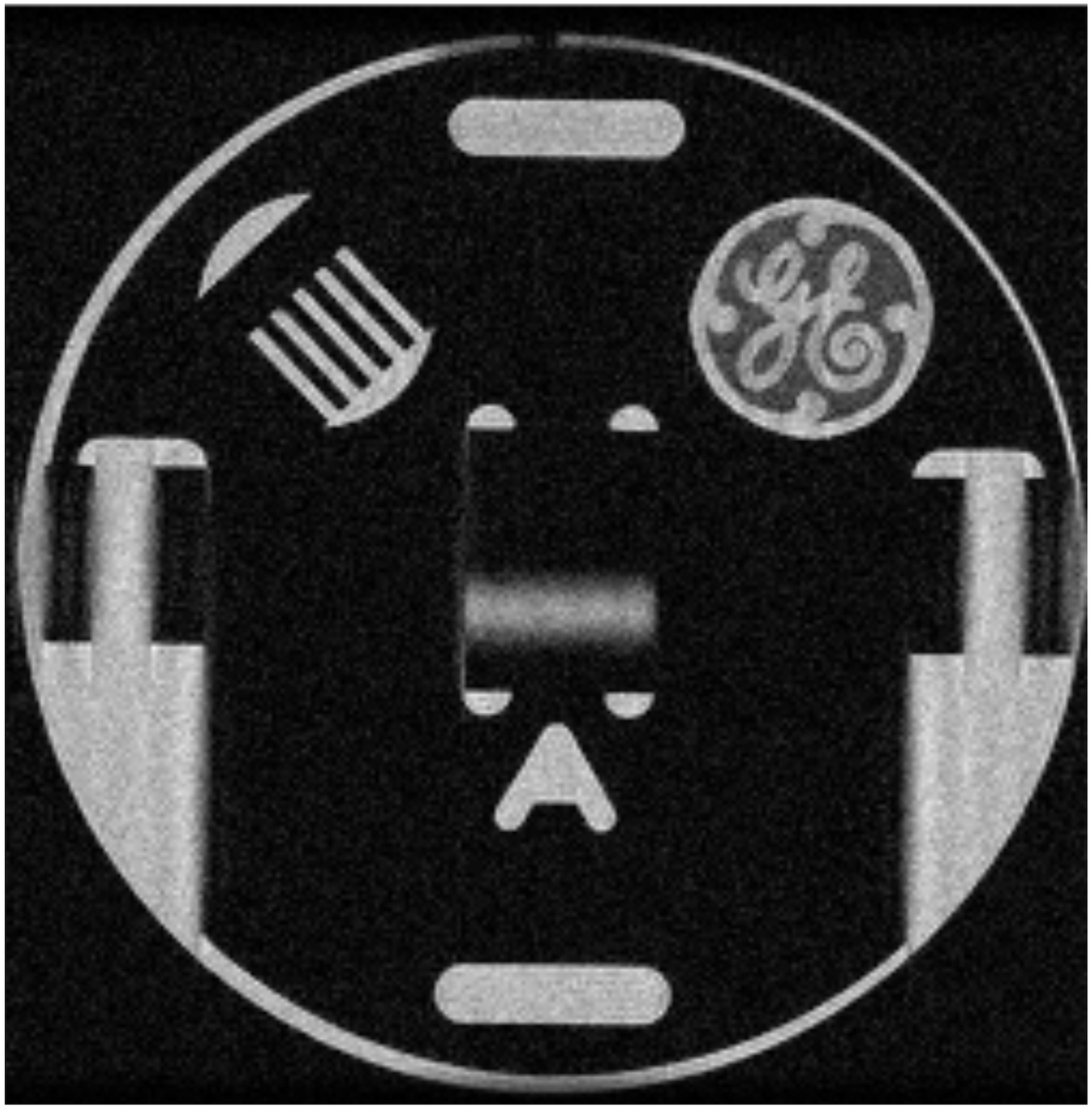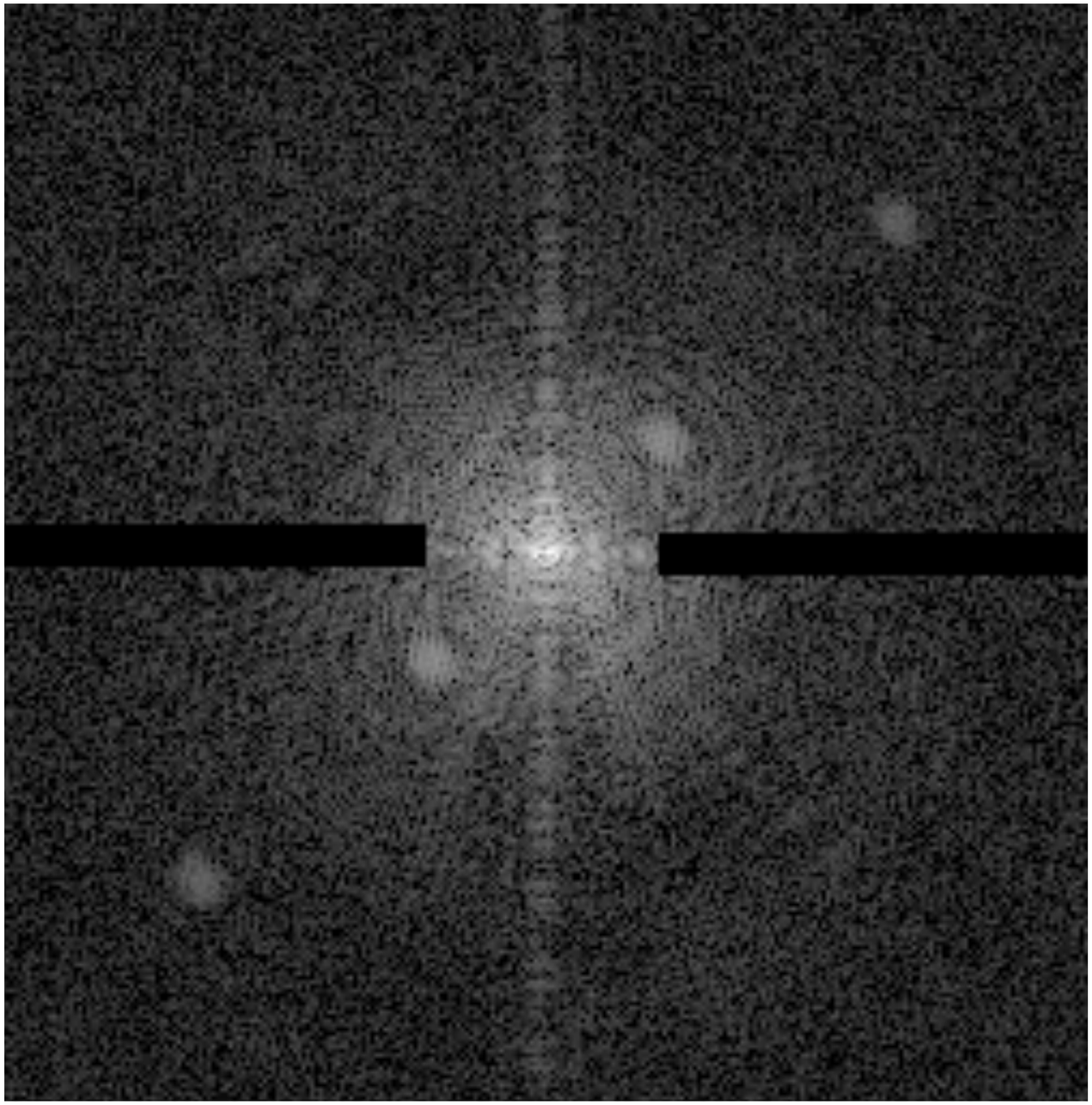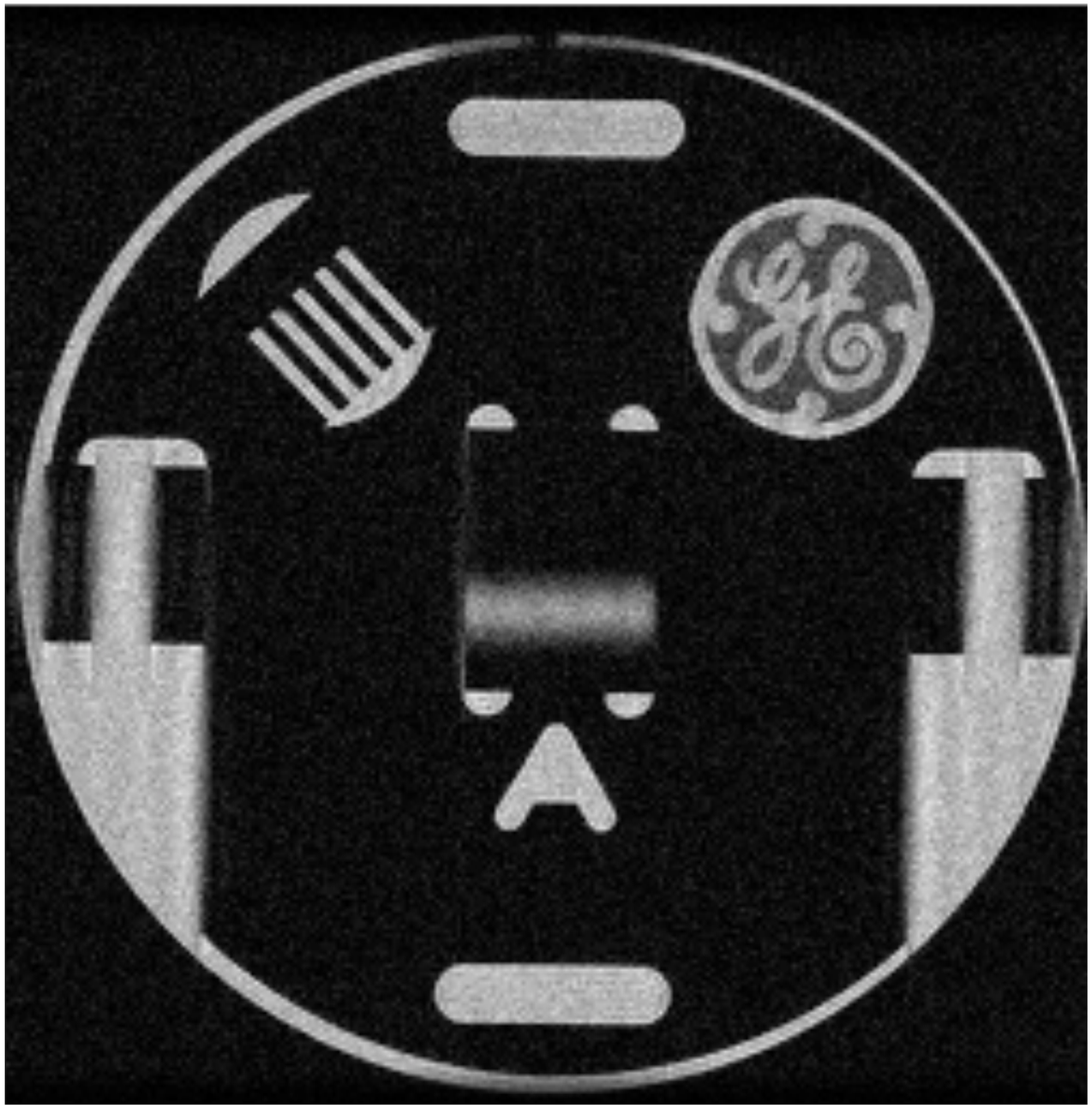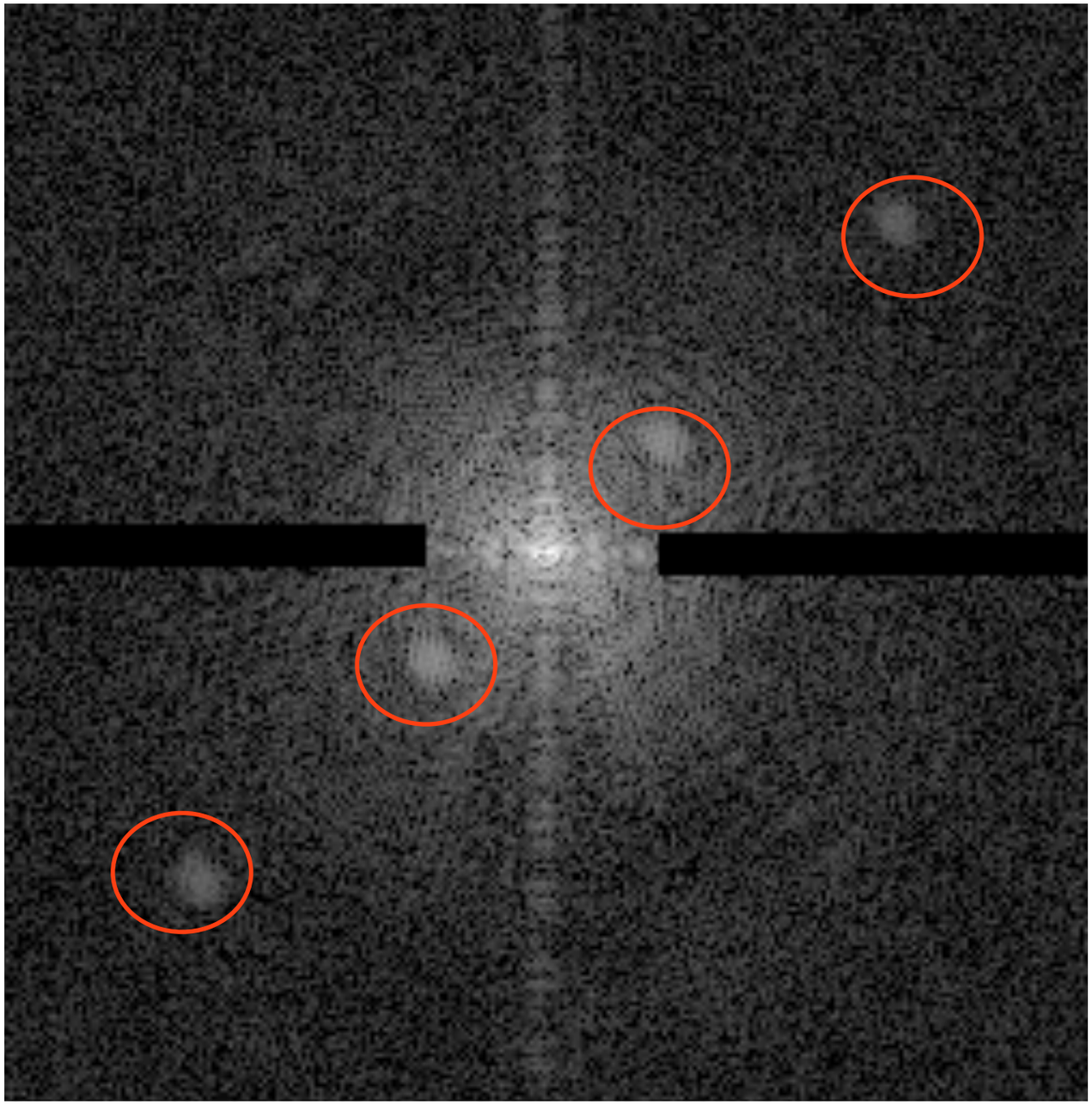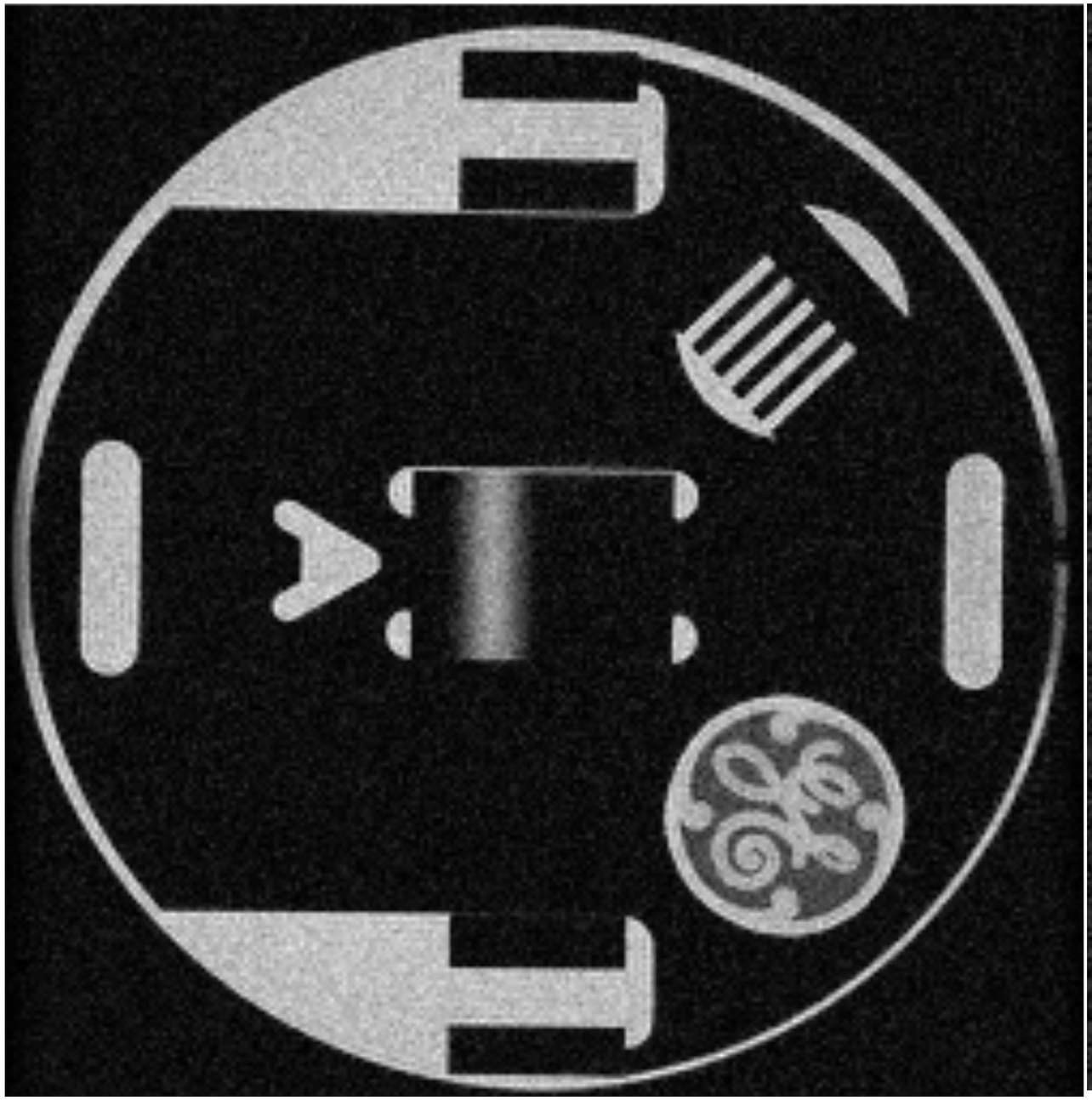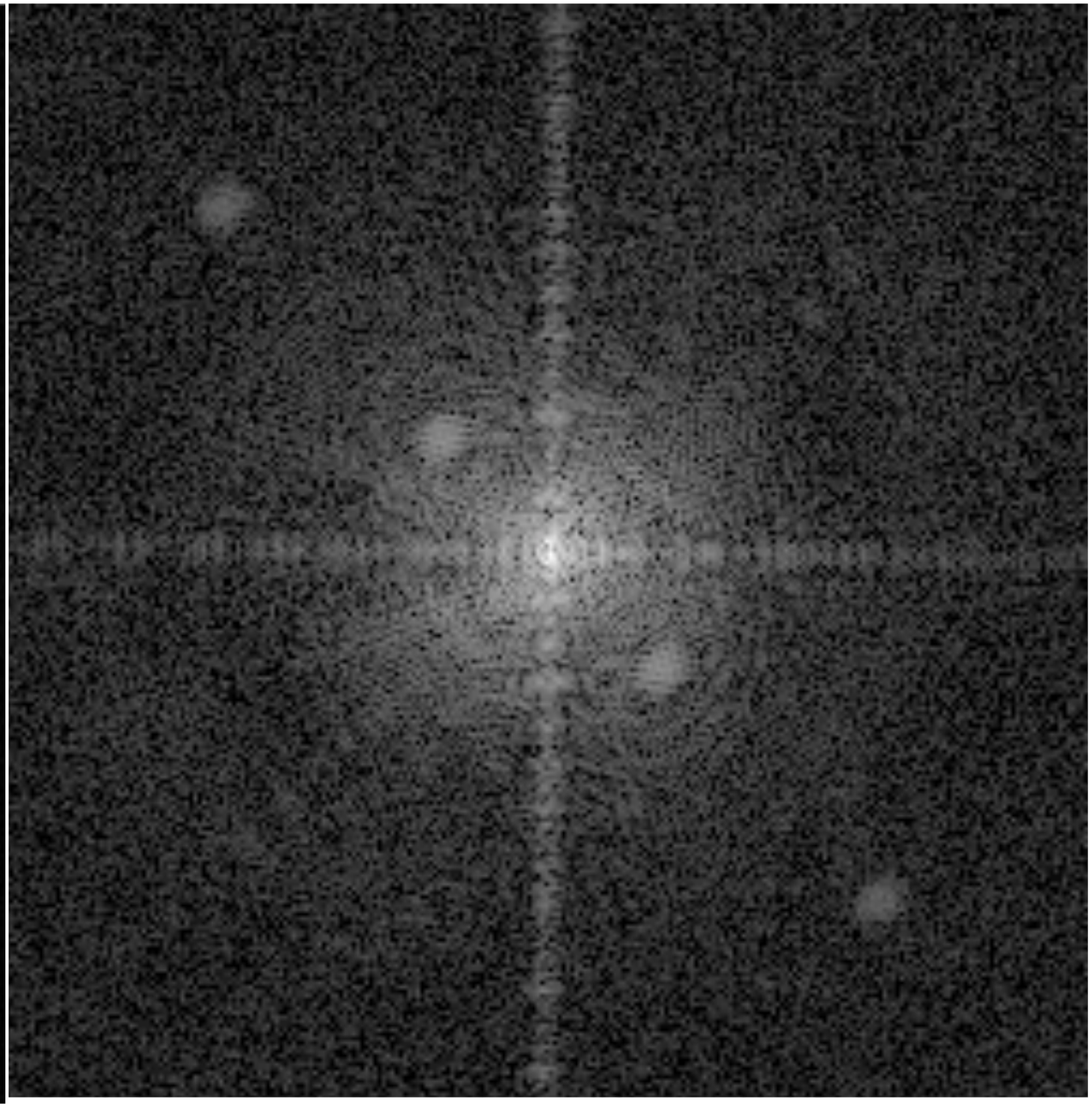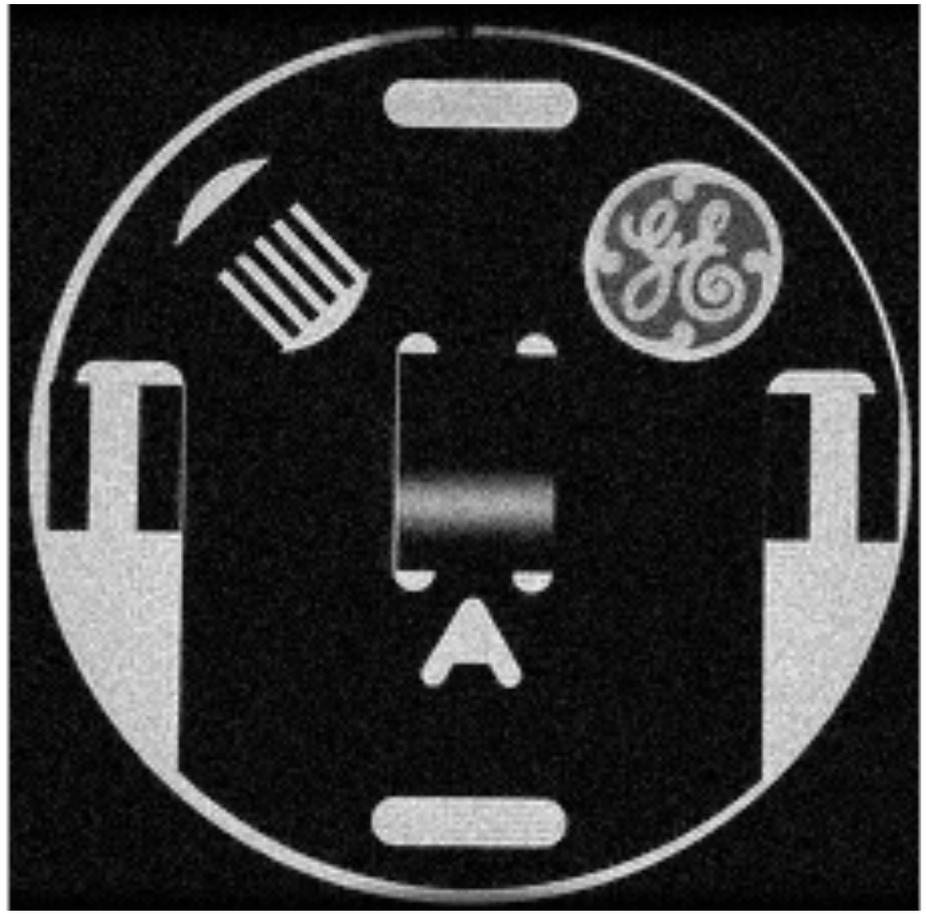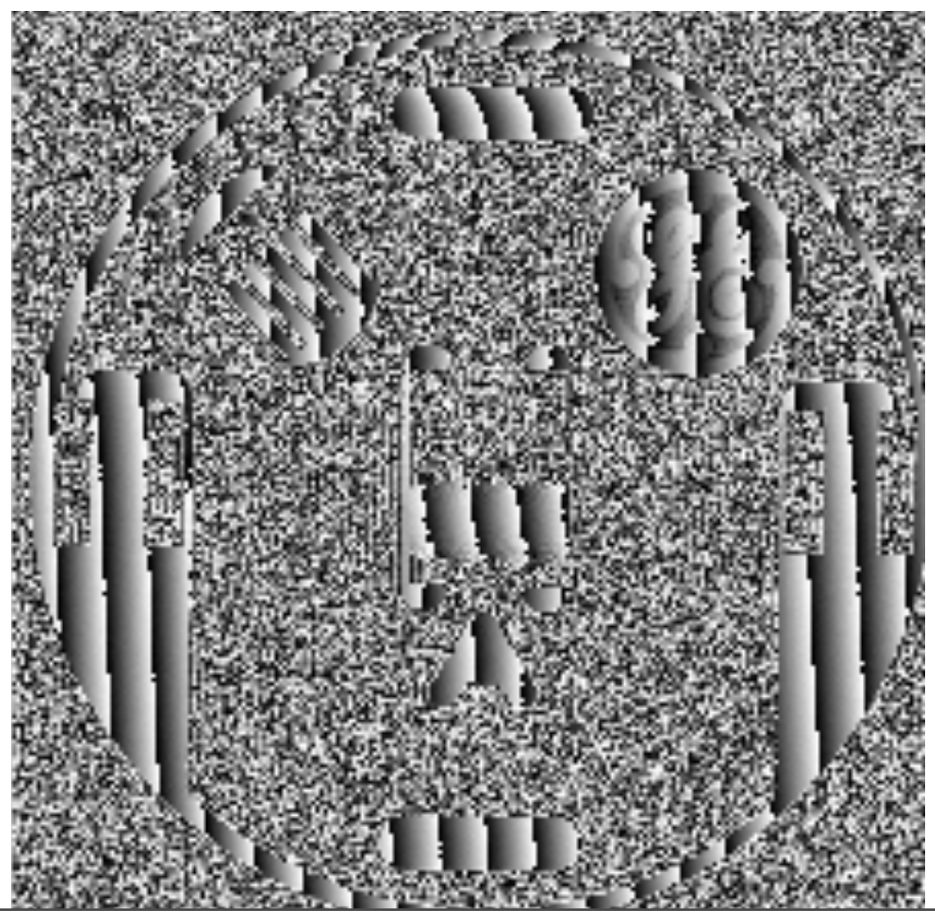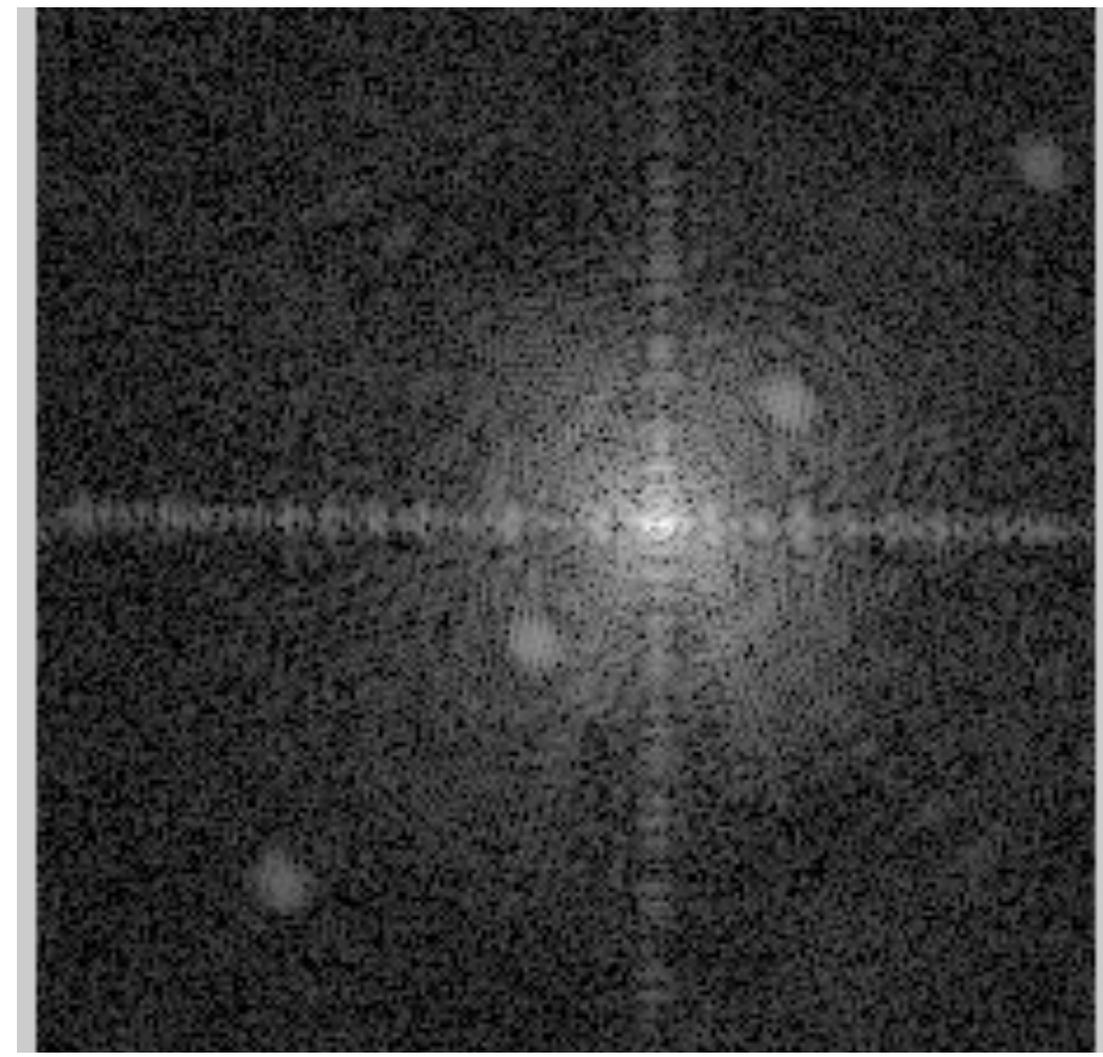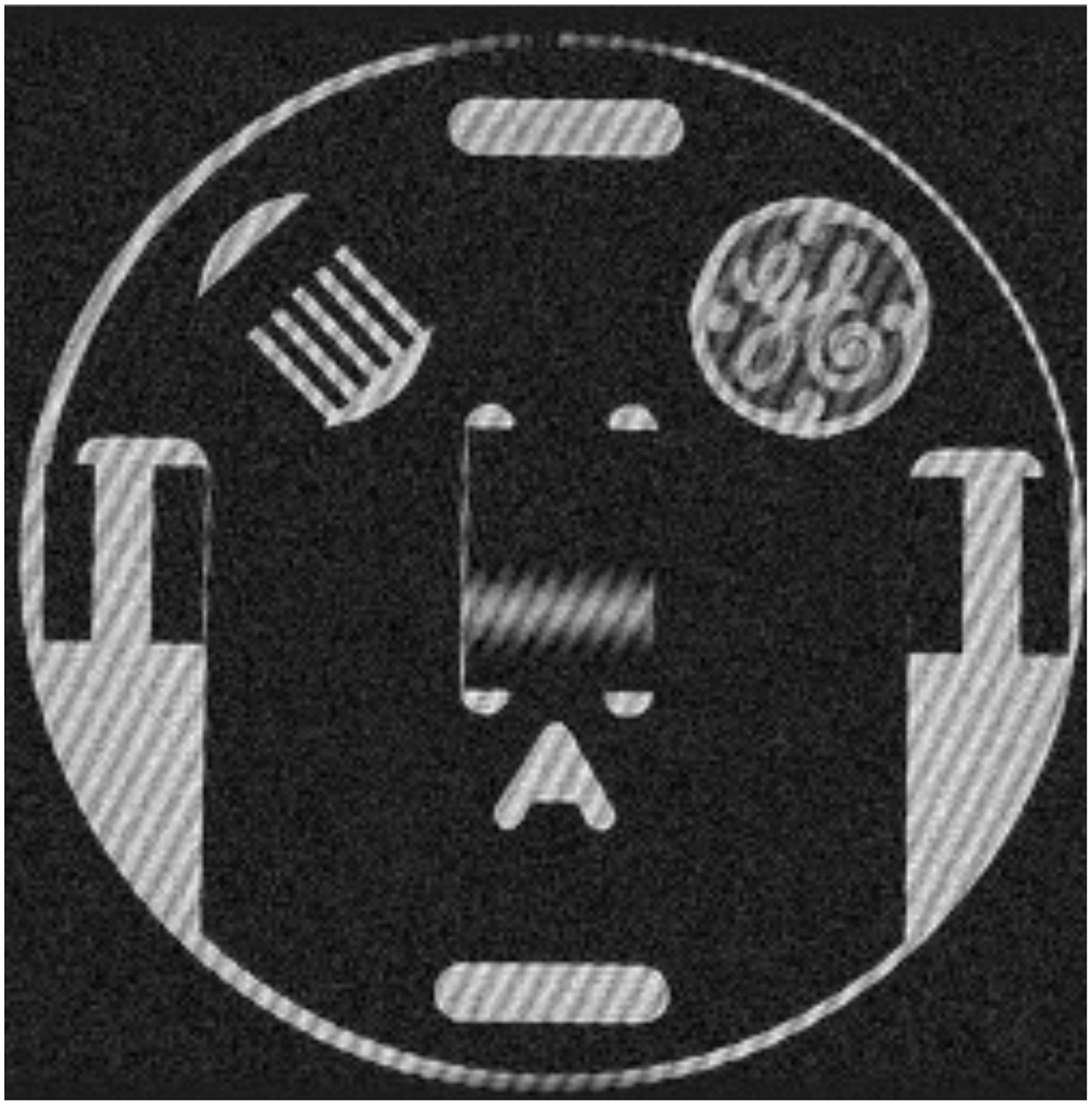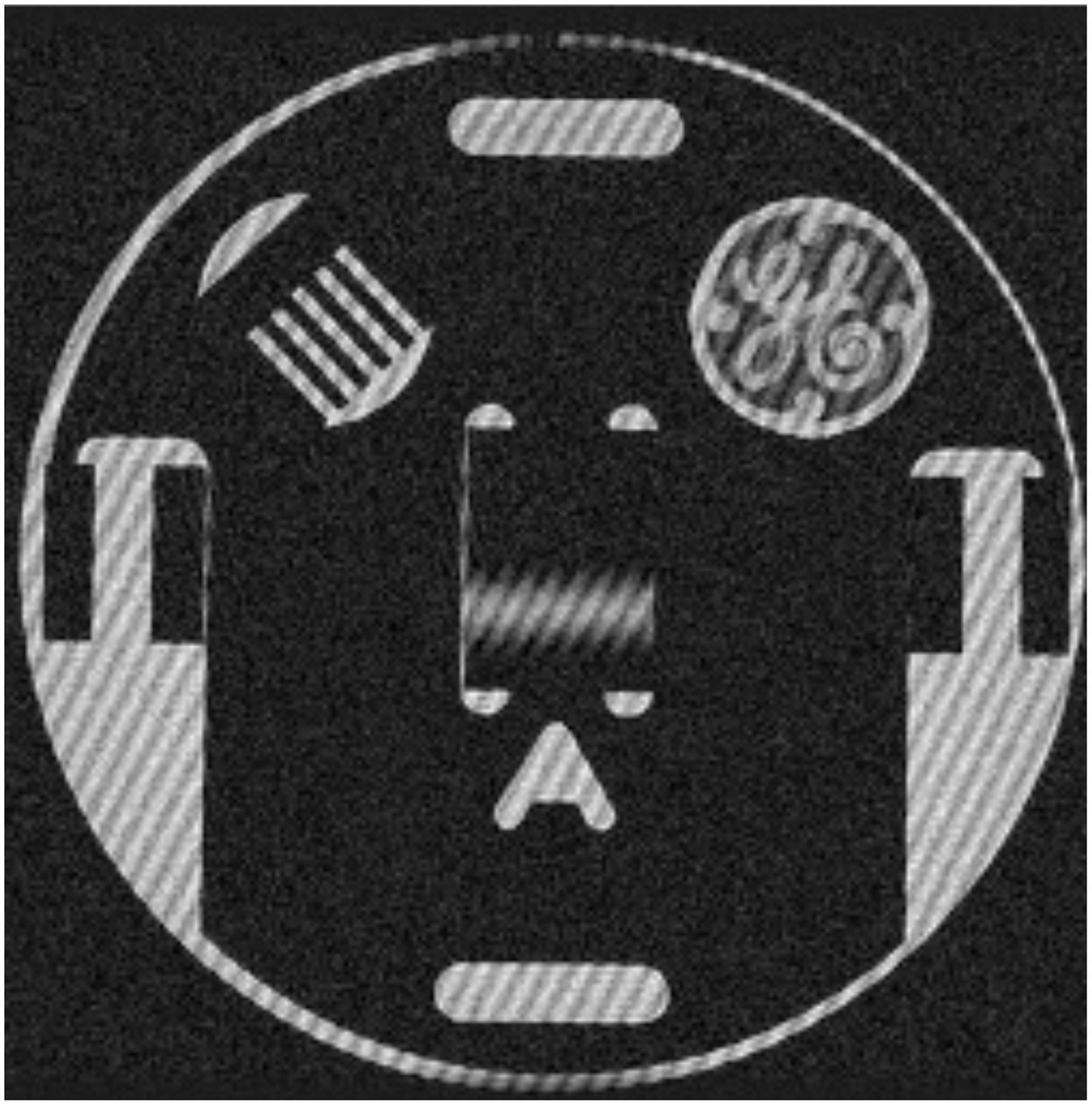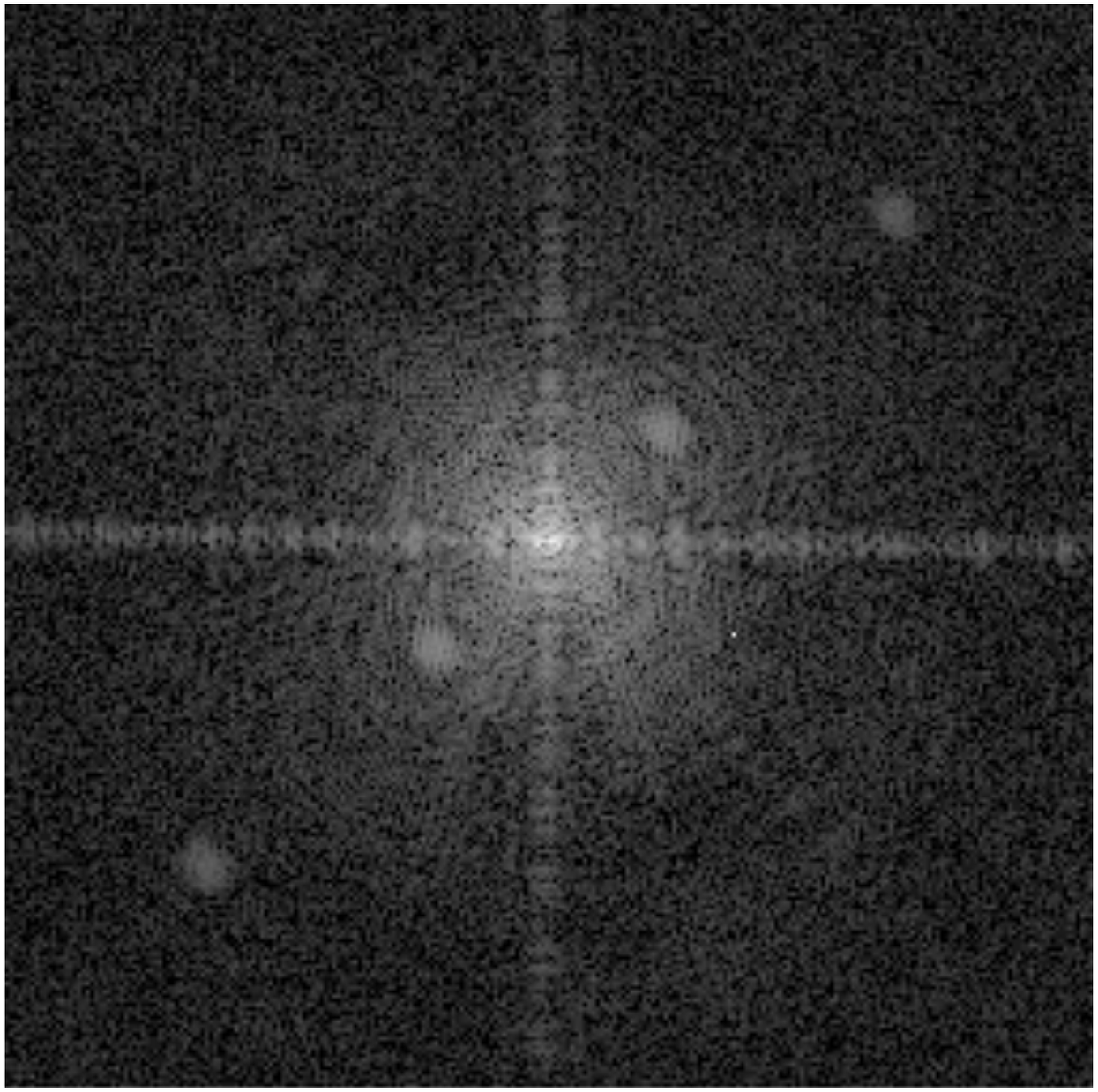
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