Discrete Time Signals

- Samples of a CT signal:
  \[ x[n] = X_a(nT) \quad n = 1, 2, \ldots \]

- Or, inherently discrete (Examples?)

## Basic Sequences

- **Unit Impulse**
  \[ \delta[n] = \begin{cases} 
  1 & n = 0 \\ 
  0 & n \neq 0 
\end{cases} \]

- **Unit Step**
  \[ U[n] = \begin{cases} 
  1 & n \geq 0 \\ 
  0 & n < 0 
\end{cases} \]

- **Exponential**
  \[ x[n] = \begin{cases} 
  A\alpha^n & n \geq 0 \\ 
  0 & n < 0 
\end{cases} \]

- For \( \alpha > 1 \) the sequence is unbounded.
- For \( -1 < \alpha < 0 \) the sequence is bounded.
- For \( 0 < \alpha < 1 \) the sequence is bounded.
- For \( \alpha < 0 \) the sequence is unbounded.
Discrete Sinusoids

\[ x[n] = A \cos(\omega_0 n + \phi) \]

or, \( x[n] = A e^{j\omega_0 n + j\phi} \)

Q: Periodic or not? \( x[n+N] = x[n] \) for \( N \) integer

- To find fundamental period \( N \)
  - Find smallest integers \( K, N \):
    \[ \omega_0 / N = 2\pi K \]

**Example:**
\[
\begin{align*}
\cos(5/7\pi n) & \quad N = 14 \quad (K = 5) \\
\cos(\pi/5n) & \quad N = 10 \quad (K = 1)
\end{align*}
\]

\[
\cos(5/7\pi n) + \cos(\pi/5n) \Rightarrow N = \text{S.C.M}\{10, 14\} = 70
\]

Q: Which one is a higher frequency?

\[ \omega_0 = \pi \quad \text{or} \quad \omega_0 = \frac{3\pi}{2} \]
Discrete Sinusoids

• Another Difference:

Q: Which one is a higher frequency?

\[ \omega_0 = \pi \quad \text{or} \quad \omega_0 = \frac{3\pi}{2} \]

\[ \cos(\pi n) \quad \cos(3\pi/2n) = \cos(\pi/2n) \]

\[ N = 2 \quad n \quad \ldots \]

\[ N = 4 \quad n \quad \ldots \]

Discrete Sinusoids

• Another Difference:

Q: Which one is a higher frequency?

\[ \omega_0 = \pi \quad \text{or} \quad \omega_0 = \frac{3\pi}{2} \]

A: \[ \omega_0 = \pi \]

Discrete Sinusoids

• Recall the periodicity of DTFT

\[ \cos(\omega_0 n) \]

\[ \omega_0 = 0 \quad \omega_0 = \pi/8 \quad \omega_0 = \pi/4 \quad \omega_0 = \pi \]
**Discrete Sinusoids**

\[ \cos(w_0 n) \]

\[ \omega_0 = 2\pi \]
\[ \omega_0 = \frac{15}{8}\pi \]
\[ \omega_0 = \frac{7}{4}\pi \]
\[ \omega_0 = \pi \]

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**Discrete Time Systems**

\[ x[n] \rightarrow T\{ \cdot \} \rightarrow y[n] \]

What properties?

- **Causality:**
  - \( y[n_0] \) depends only on \( x[n] \) for \( \infty \leq n \leq n_0 \)

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**Properties of D.T. Systems Cont.**

- **Memoryless:**
  - \( y[n] \) depends only on \( x[n] \)
  - Example: \( y[n] = x[n]^2 \)

- **Linearity:**
  - **Superposition:**
    \[ T\{ x_1[n] + x_2[n] \} = T\{ x_1[n] \} + T\{ x_2[n] \} = y_1[n] + y_2[n] \]
  - **Homogeneity:**
    \[ T\{ ax_1[n] \} = aT\{ x[n] \} = ay[n] \]

- **Time Invariance:**
  - If: \( y[n] = T\{ x[n] \} \)
  - Then: \( y[n-n_0] = T\{ x[n-n_0] \} \)

- **BIBO Stability**
  - If: \( |x[n]| \leq B_x < \infty \quad \forall \ n \)
  - Then: \( |y[n]| \leq B_y < \infty \quad \forall \ n \)
### Example:

<table>
<thead>
<tr>
<th></th>
<th>Causal</th>
<th>L</th>
<th>TI</th>
<th>memoryless</th>
<th>BIBO stable</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time Shift</strong></td>
<td>if $nd&gt;0$</td>
<td>Y</td>
<td>Y</td>
<td>if $nd=0$</td>
<td>Y</td>
</tr>
<tr>
<td>$y[n] = x[n-n_d]$</td>
<td></td>
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<tr>
<td><strong>Accumulator</strong></td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>$y[n] = \sum_{k=-\infty}^{n} x[k]$</td>
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</tr>
<tr>
<td><strong>Compressor</strong></td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>$y[n] = x[Mn]$</td>
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<tr>
<td>$M &gt; 1$</td>
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</tbody>
</table>

### Examples

**Non-Linear system: Median Filter**

$$y[n] = \text{MED}\{x[n-k], \ldots, x[n+k]\}$$

### Example: Removing Shot Noise

- From NPR This American Life, ep.203
  - "The Greatest Phone Message in the World"

  "em.. (giggle) There comes a time in life, when.. eh... when we hear the greatest phone mail message of all times and .... well here it is... eh.. you have to hear it to believe it."

  corrupted message
Spectrum of Speech

Speech

Corrupted Speech

Low Pass Filtering

LP-Filter Spectrum

Low-Pass Filtering of Shot Noise

Corrupted

LP-filtered

Med-filter
The Greatest Message of All Times...

- I thought you’ll get a kick out of a message from my mother......
  Hi Fred, you and the little mermaid can go blip yourself. I told you to stay near the phone... I can’t find those books.. you have other books here.. it must be in Le’hoya.. call me back... I’m not going to stay up all night for you... .... Bye Bye...

Discrete-Time LTI Systems

- The impulse response $h[n]$ completely characterizes an LTI system “DNA of LTI”

\[ \delta[n] \xrightarrow{\text{LTI}} h[n] \]
\[ x[n] \xrightarrow{\text{LTI}} y[n] = h[n] \ast x[n] \]
\[ y[n] = \sum_{N=-\infty}^{\infty} h[m] x[n-m] \]

Sum of weighted, delayed impulse responses!

BIBO Stability of LTI Systems

- An LTI system is BIBO stable iff $h[n]$ is absolutely summable

\[ \sum_{k=-\infty}^{\infty} |h[k]| < \infty \]

BIBO Stability of LTI Systems

- Proof: “if”

\[ |y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| \cdot |x[n-k]| \leq B_x \]
\[ \leq B_x \sum_{k=-\infty}^{\infty} |h[k]| < \infty \]
BIBO Stability of LTI Systems

• Proof: “only if”
  - Suppose $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$
  - Show that there exists bounded $x[n]$ that gives unbounded $y[n]$

- Let: $x[n] = \frac{h[-n]}{|h[-n]|} = \text{Sign}\{h[-n]\}$

  
  \[
y[n] = \sum h[k]x[n-k]
  \]

  
  \[
y[0] = \sum h[k]x[-k] = \sum h[k]h[k]/|h[k]| = \sum |h[k]| = \infty
  \]