2.42. (a) From the figure,

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= (x[n] * (\delta[n] + h_1[n])) * h_2[n]. \end{aligned}$$

Let $h[n]$ be the impulse response of the overall system,

$$y[n] = x[n] * h[n].$$

Comparing with the above expression,

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= h_2[n] + h_1[n] * h_2[n] \\ &= \alpha^n u[n] + \beta^{(n-1)} u[n-1]. \end{aligned}$$

(b) Taking the Fourier transform of $h[n]$ from part (a),

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} + \beta \sum_{n=-\infty}^{\infty} \alpha^{(n-1)} u[n-1] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} + \beta \sum_{\ell=0}^{\infty} \alpha^{(\ell-1)} e^{-j\omega \ell}, \end{aligned}$$

where we have used $\ell = (n-1)$ in the second sum.

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{1-\alpha e^{-j\omega}} + \frac{\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \text{ for } |\alpha| < 1. \end{aligned}$$

Note that the Fourier transform of $\alpha^n u[n]$ is well known, and the second term of $h[n]$ (see part (a)) is just a scaled and shifted version of $\alpha^n u[n]$. So, we could have used the properties of the Fourier transform to reduce the algebra.

(c) We have

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1-\alpha e^{-j\omega}] = X(e^{j\omega})[1+\beta e^{-j\omega}]$$

taking the inverse Fourier transform, we have

$$y[n] - \alpha y[n-1] = x[n] + \beta x[n-1].$$

(d) From part (a):

$$h[n] = 0, \text{ for } n < 0.$$

This implies that the system is CAUSAL.

If the system is stable, its Fourier transform exists. Therefore, the condition for stability is the same as the condition imposed on the frequency response of part (b). That is, STABLE, if $|\alpha| < 1$.

2.56. (a)

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n} x[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k} x[k] h[n-k]. \end{aligned}$$

Let $x[n] = ax_1[n] + bx_2[n]$, then:

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n}(ax_1[n] + bx_2[n])) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k}(ax_1[k] + bx_2[k]) h[n-k] \\ &= a \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k} x_1[k] h[n-k] + b \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k} x_2[k] h[n-k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

where $y_1[n]$ and $y_2[n]$ are the responses to $x_1[n]$ and $x_2[n]$ respectively. We thus conclude that system S is linear.

(b) Let $x_2[n] = x[n - n_0]$, then:

$$\begin{aligned} y_2[n] &= h[n] * (e^{-j\omega_0 n} x_2[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x_2[n-k] h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x[n - n_0 - k] h[k] \\ &\neq y[n - n_0]. \end{aligned}$$

We thus conclude that system S is not time invariant.

(c) Since the magnitude of $e^{-j\omega_0 n}$ is always bounded by 1 and $h[n]$ is stable, a bounded input $x[n]$ will always produce a bounded input to the stable LTI system and therefore the output $y[n]$ will be bounded. We thus conclude that system S is stable.

(d) We can rewrite $y[n]$ as:

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n} x[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x[n-k] h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 n} e^{j\omega_0 k} x[n-k] h[k] \\ &= e^{-j\omega_0 n} \sum_{k=-\infty}^{+\infty} e^{j\omega_0 k} x[n-k] h[k]. \end{aligned}$$

System C should therefore be a multiplication by $e^{-j\omega_0 n}$.

2.59. (a) Using the change of variable: $r = -k$, we can rewrite $R_x[n]$ as:

$$R_x[n] = \sum_{r=-\infty}^{\infty} x^*[-r]x[n-r] = x^*[-n] * x[n].$$

We therefore have:

$$g[n] = x^*[-n].$$

(b) The Fourier transform of $x^*[-n]$ is $X^*(e^{j\omega})$, therefore:

$$R_x(e^{j\omega}) = X^*(e^{j\omega})X(e^{j\omega}) = |X(e^{j\omega})|^2.$$

2.60. (a) Note that $x_2[n] = -\sum_{k=0}^{k=4} x[n-k]$. Since the system is LTI, we have:

$$y_2[n] = -\sum_{k=0}^{k=4} y[n-k].$$

(b) By carrying out the convolution, we get:

$$h[n] = \begin{cases} -1 & , \quad n = 0, n = 2 \\ -2 & , \quad n = 1 \\ 0 & , \quad \text{o.w.} \end{cases}$$

3.28. (a)

$$nx[n] \Leftrightarrow -z\frac{d}{dx}X(z)$$

$$x[n-n_0] \Leftrightarrow z^{-n_0}X(z)$$

$$X(z) = \frac{3z^{-3}}{(1-\frac{1}{4}z^{-1})^2} = 12z^{-2}\left[-z\frac{d}{dz}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\right]$$

$x[n]$ is left-sided. Therefore, $X(z)$ corresponds to:

$$x[n] = -12(n-2)\left(\frac{1}{4}\right)^{n-2} u[-n+1]$$

(b)

$$X(z) = \sin(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{2k+1} \qquad \text{ROC includes } |z| = 1$$

Therefore,

$$x[n] = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\delta[n+2k+1]$$

Which is stable.

(c)

$$X(z) = \frac{z^7-2}{1-z^{-7}} = z^7 - \frac{1}{1-z^{-7}} \qquad |z| > 1$$

$$X(z) = z^7 - \sum_{n=0}^{\infty} z^{-7n}$$

Therefore,

$$x[n] = \delta[n+7] - \sum_{n=0}^{\infty}\delta[n-7k]$$

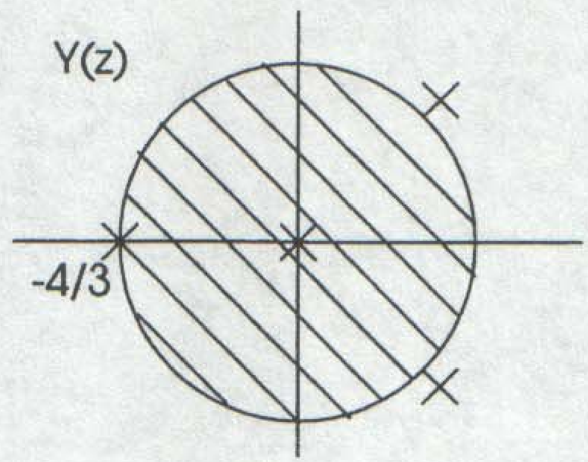
3.32. From the pole-zero diagram

$$X(z) = \frac{z}{(z^2 - z + \frac{1}{2})(z + \frac{3}{4})} \qquad |z| > \frac{3}{4}$$

$$\begin{aligned} y[n] &= x[-n+3] = x[-(n-3)] \\ \Rightarrow \quad & Y(z) = z^{-3}X(z^{-1}) = \frac{z^{-3}z^{-1}}{(z^{-2} - z^{-1} + \frac{1}{2})(z^{-1} + \frac{3}{4})} \\ &= \frac{8/3}{z(2 - 2z + z^2)(\frac{4}{3} + z)} \end{aligned}$$

Poles at $0, -\frac{4}{3}, 1 \pm j$, zeros at ∞

$x[n]$ causal $\Rightarrow x[-n+3]$ is left-sided $\Rightarrow$ ROC is $0 < |z| < 4/3$.



3.38.

$$Y(z) = \frac{z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} \cdot \frac{2}{1 - z^{-1}} \qquad |z| > 1$$

Therefore using a contour C that lies outside of $|z| = 1$ we get

$$\begin{aligned} y[1] &= \frac{1}{2\pi j} \oint_C \frac{2(z+1)z^n dz}{(z - \frac{1}{2})(z + \frac{1}{3})(z - 1)} \\ &= \frac{2(\frac{1}{2}+1)(\frac{1}{2})}{(\frac{1}{2}+\frac{1}{3})(\frac{1}{2}-1)} + \frac{2(-\frac{1}{3}+1)(-\frac{1}{3})}{(-\frac{1}{3}-\frac{1}{2})(-\frac{1}{3}-1)} + \frac{2(1+1)(1)}{(1-\frac{1}{2})(1+\frac{1}{3})} \\ &= -\frac{18}{5} - \frac{2}{5} + 6 = 2 \end{aligned}$$

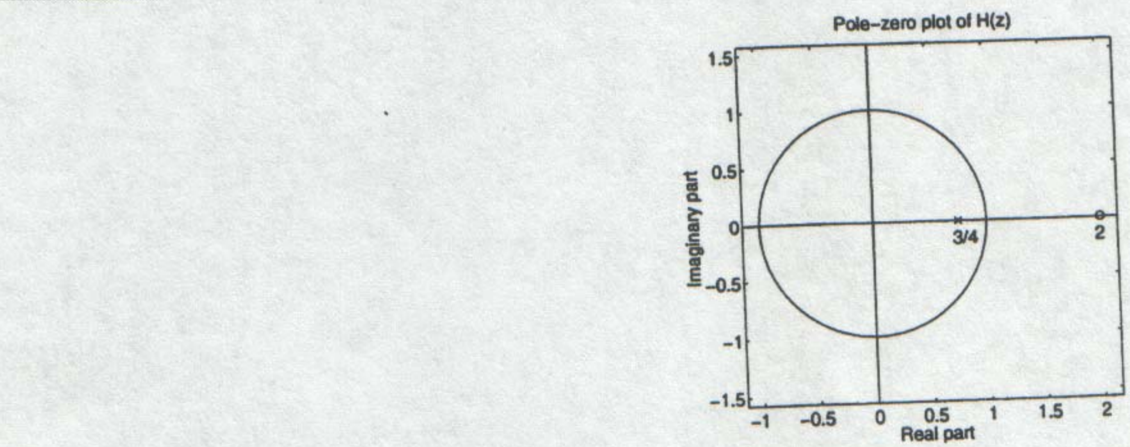
3.43. (a)

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}}, \qquad \frac{1}{2} < |z| < 2$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})}}{\frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}}$$

$$= \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$



(b)

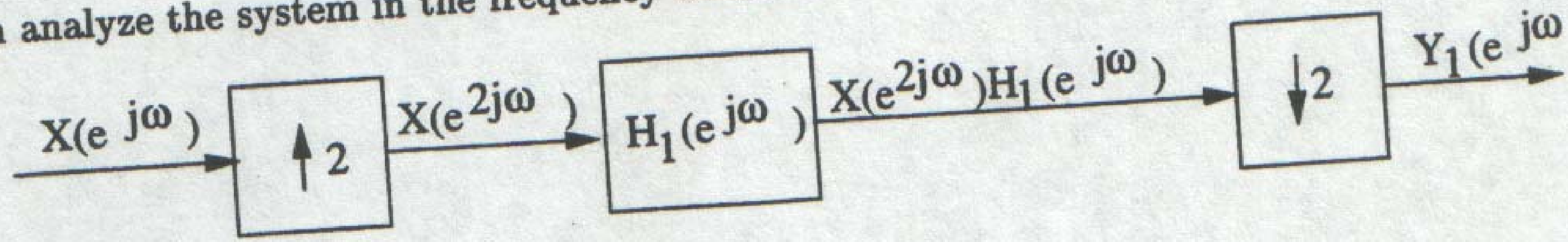
$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

(c)

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$$

(d) The system is stable because the ROC includes the unit circle. It is also causal since $h[n] = 0$ for $n < 0$.

4.29. We can analyze the system in the frequency domain:



$Y_1(e^{j\omega})$ is $X(e^{2j\omega})H_1(e^{j\omega})$ downsampled by 2:

$$
\begin{aligned}
Y_1(e^{j\omega}) &= \frac{1}{2}\left\{X(e^{2j\omega/2})H_1(e^{j\omega/2}) + X(e^{(2j(\omega-2\pi)/2})H_1(e^{j(\omega-2\pi)/2})\right\} \\
&= \frac{1}{2}\left\{X(e^{j\omega})H_1(e^{j\omega/2}) + X(e^{j(\omega-2\pi)})H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} \\
&= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} X(e^{j\omega}) \\
&= H_2(e^{j\omega})X(e^{j\omega}) \\
H_2(e^{j\omega}) &= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\}
\end{aligned}
$$

4.30.

$$\begin{array}{ll} X_c(j\Omega) = 0 & |\Omega| \geq 4000\pi \\ Y(j\Omega) = |\Omega| X_c(j\Omega), & 1000\pi \leq |\Omega| \leq 2000\pi \end{array}$$

Since only half the frequency band of $X_c(j\Omega)$ is needed, we can alias everything past $\Omega = 2000\pi$. Hence, $T = 1/3000$ s.

Now that T is set, figure out $H(e^{j\omega})$ band edges.

$$\omega_1 = \Omega_1 T \quad \Rightarrow \omega_1 = 2\pi \cdot 500 \cdot \tfrac{1}{3000} \quad \Rightarrow \omega_1 = \frac{\pi}{3}$$

$$\omega_2 = \Omega_2 T \quad \Rightarrow \omega_2 = 2\pi \cdot 1000 \cdot \tfrac{1}{3000} \quad \Rightarrow \omega_2 = \frac{2\pi}{3}$$

$$H(e^{j\omega}) = \begin{cases} |\omega| & \frac{\pi}{3} \leq |\omega| \leq \frac{2\pi}{3} \\ 0 & 0 \leq |\omega| < \frac{\pi}{3}, \frac{2\pi}{3} < |\omega| \leq \pi \end{cases}$$

4.34. (a) Since there is no aliasing involved in this process, we may choose T to be any value. Choose $T = 1$ for simplicity. $X_c(j\Omega) = 0, |\Omega| \geq \pi/T$. Since $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$, $Y_c(j\Omega) = 0, |\Omega| \geq \pi/T$. Therefore, there will be no aliasing problems in going from $y_c(t)$ to $y[n]$.
Recall the relationship $\omega = \Omega T$. We can simply use this in our system conversion:

$$\begin{aligned} H(e^{j\omega}) &= e^{-j\omega/2} \\ H(j\Omega) &= e^{-j\Omega T/2} \\ &= e^{-j\Omega/2}, \quad T = 1 \end{aligned}$$

Note that the choice of T and therefore $H(j\Omega)$ is not unique.

(b)

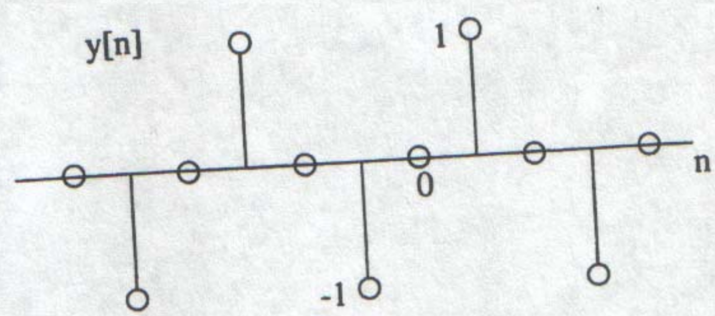
$$\begin{aligned} \cos\left(\frac{5\pi}{2}n - \frac{\pi}{4}\right) &= \frac{1}{2}\left[e^{j(\frac{5\pi}{2}n-\frac{\pi}{4})} + e^{-j(\frac{5\pi}{2}n-\frac{\pi}{4})}\right] \\ &= \frac{1}{2}e^{-j(\pi/4)}e^{j(5\pi/2)n} + \frac{1}{2}e^{j(\pi/4)}e^{-j(5\pi/2)n} \end{aligned}$$

Since $H(e^{j\omega})$ is an LTI system, we can find the response to each of the two eigenfunctions separately.

$$y[n] = \frac{1}{2}e^{-j(\pi/4)}H\left(e^{j(5\pi/2)}\right)e^{j(5\pi/2)n} + \frac{1}{2}e^{j(\pi/4)}H\left(e^{-j(5\pi/2)}\right)e^{-j(5\pi/2)n}$$

Since $H(e^{j\omega})$ is defined for $0 \leq |\omega| \leq \pi$ we must evaluate the frequency at the baseband, i.e., $5\pi/2 \Rightarrow 5\pi/2 - 2\pi = \pi/2$. Therefore,

$$\begin{aligned} y[n] &= \frac{1}{2}e^{-j(\pi/4)}H\left(e^{j(5\pi/2)}\right)e^{j(5\pi/2)n} + \frac{1}{2}e^{j(\pi/4)}H\left(e^{-j(5\pi/2)}\right)e^{-j(5\pi/2)n} \\ &= \frac{1}{2}\left(e^{j[(5\pi/2)n-(\pi/2)]} + e^{-j[(5\pi/2)n-(\pi/2)]}\right) \\ &= \cos\left(\frac{5\pi}{2}n - \frac{\pi}{2}\right). \end{aligned}$$



4.37. In both systems, the speech was filtered first so that the subsequent sampling results in no aliasing. Therefore, going $s[n]$ to $s_1[n]$ basically requires changing the sampling rate by a factor of 3kHz/5kHz = 3/5. This is done with the following system: