7.25. (a) **Answer:** Only the bilinear transform design will guarantee that a minimum phase discrete-time filter is created from a minimum phase continuous-time filter. For the following explanations remember that a discrete-time minimum phase system has all its poles and zeros inside the unit circle.

Impulse Invariance: Impulse invariance maps left-half s-plane poles to the interior of the z-plane unit circle. However, left-half s-plane zeros will *not necessarily* be mapped inside the z-plane unit circle. Consider:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} = \frac{\sum_{k=1}^{N} A_k \prod_{\substack{j=1\\ j\neq k}}^{N} (s - s_j)}{\prod_{\ell=1}^{N} (s - s_\ell)}$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}} = \frac{\sum_{k=1}^{N} T_d A_k \prod_{\substack{j=1\\ j\neq k}}^{N} (1 - e^{s_j T_d} z^{-1})}{\prod_{\ell=1}^{N} (1 - e^{s_\ell T_d} z^{-1})}$$

If we define $\text{Poly}_k(z) = \prod_{\substack{j=1\\ j\neq k}}^{N} \left(1 - e^{s_j T_d} z^{-1}\right)$, we can note that all the roots of $\text{Poly}_k(z)$ are inside the unit circle. Since the numerator of $H(z)$ is a sum of $A_k\text{Poly}_k(z)$ terms, we see that there are *no guarantees* that the roots of the numerator polynomial are inside the unit circle. In other words, the sum of minimum phase filters is not necessarily minimum phase. By considering the specific example of

$$H_c(s) = \frac{s + 10}{(s + 1)(s + 2)},$$

and using $T = 1$, we can show that a minimum phase filter is transformed into a non-minimum phase discrete time filter.

Bilinear Transform: The bilinear transform maps a pole or zero at $s = s_0$ to a pole or zero (respectively) at $z_0 = \frac{1+\frac{T}{2}s_0}{1-\frac{T}{2}s_0}$. Thus,

$$|z_0| = \left|\frac{1 + \frac{T}{2}s_0}{1 - \frac{T}{2}s_0}\right|$$

Since $H_c(s)$ is minimum phase, all the poles of $H_c(s)$ are located in the left half of the s-plane. Therefore, a pole $s_0 = \sigma + j\Omega$ must have $\sigma < 0$. Using the relation for s_0, we get

$$\begin{aligned} |z_0| &= \sqrt{\frac{(1 + \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}{(1 - \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}} \\ &< 1 \end{aligned}$$

Thus, all poles and zeros will be inside the z-plane unit circle and the discrete-time filter will be minimum phase as well.

(b) **Answer:** Only the bilinear transform design will result in an allpass filter.

Impulse Invariance: In the impulse invariance design we have

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)$$

The aliasing terms can destroy the allpass nature of the continuous-time filter.

Bilinear Transform: The bilinear transform only warps the frequency axis. The magnitude response is not affected. Therefore, an allpass filter will map to an allpass filter.

(c) **Answer:** Only the bilinear transform will guarantee

$$H(e^{j\omega})|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

Impulse Invariance: Since impulse invariance may result in aliasing, we see that

$$H(e^{j0}) = H_c(j0)$$

if and only if

$$H(e^{j0}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = H_c(j0)$$

or equivalently

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0$$

which is generally not the case.

Bilinear Transform: Since, under the bilinear transformation, $\Omega = 0$ maps to $\omega = 0$,

$$H(e^{j0}) = H_c(j0)$$

for all $H_c(s)$.

(d) **Answer:** Only the bilinear transform design is guaranteed to create a bandstop filter from a bandstop filter.

If $H_c(s)$ is a bandstop filter, the bilinear transform will preserve this because it just warps the frequency axis; however aliasing (in the impulse invariance technique) can fill in the stop band.

(e) **Answer:** The property holds under the bilinear transform, but not under impulse invariance.

Impulse Invariance: Impulse invariance may result in aliasing. Since the order of aliasing and multiplication are not interchangeable, the desired identity does not hold. Consider $H_{a_1}(s) = H_{a_2}(s) = e^{-sT/2}$.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned} H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &\equiv H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_1(z)H_2(z) \end{aligned}$$

(f) **Answer:** The property holds for both impulse invariance and the bilinear transform.

Impulse Invariance:

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d}+\frac{2\pi}{T_d}k\right)\right) \\
&= \sum_{k=-\infty}^{\infty} H_{c1}\left(j\left(\frac{\omega}{T_d}+\frac{2\pi}{T_d}k\right)\right) + \sum_{k=-\infty}^{\infty} H_{c2}\left(j\left(\frac{\omega}{T_d}+\frac{2\pi}{T_d}k\right)\right) \\
&= H_1(e^{j\omega}) + H_2(e^{j\omega})
\end{aligned}
$$

Bilinear Transform:

$$
\begin{aligned}
H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\
&= H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\
&= H_1(z) + H_2(z)
\end{aligned}
$$

(g) **Answer:** Only the bilinear transform will result in the desired relationship.

Impulse Invariance: By impulse invariance,

$$
\begin{aligned}
H_1\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_1}\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right) \\
H_2\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_2}\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right)
\end{aligned}
$$

We can clearly see that due to the aliasing, the phase relationship is not guaranteed to be maintained.

Bilinear Transform: By the bilinear transform,

$$
\begin{aligned}
H_1(e^{j\omega}) &= H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right) \\
H_2(e^{j\omega}) &= H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right)
\end{aligned}
$$

therefore,

$$
\frac{H_1(e^{j\omega})}{H_2(e^{j\omega})} = \frac{H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right)}{H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right)} = \begin{cases} e^{-j\pi/2}, & 0 < \omega < \pi \\ e^{j\pi/2}, & -\pi < \omega < 0 \end{cases}
$$

7.28. (a) We have

$$
\begin{aligned}
s &= \frac{1-z^{-1}}{1+z^{-1}} \\
j\Omega &= \frac{1-e^{-j\omega}}{1+e^{-j\omega}} \\
&= \frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}} \\
\Omega &= \tan\left(\frac{\omega}{2}\right)
\end{aligned}
$$

$$\Omega_p = \tan\left(\frac{\omega_{p_1}}{2}\right) \longleftrightarrow \omega_{p_1} = 2\tan^{-1}(\Omega_p)$$

(b)

$$
\begin{aligned}
s &= \frac{1+z^{-1}}{1-z^{-1}} \\
j\Omega &= \frac{1+e^{-j\omega}}{1-e^{-j\omega}} \\
&= \frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}} \\
\Omega &= -\cot\left(\frac{\omega}{2}\right) \\
&= \tan\left(\frac{\omega-\pi}{2}\right)
\end{aligned}
$$

$$\Omega_p = \tan\left(\frac{\omega_{p_2}-\pi}{2}\right) \longleftrightarrow \omega_{p_2} = \pi + 2\tan^{-1}(\Omega_p)$$

(c)

$$\tan\left(\frac{\omega_{p_2}-\pi}{2}\right) = \tan\left(\frac{\omega_{p_1}}{2}\right)$$

$$\Rightarrow \omega_{p_2} = \omega_{p_1} + \pi$$

(d)

$$H_2(z) = H_1(z)|_{z=-z}$$

The even powers of z do not get changed by this transformation, while the coefficients of the odd powers of z change sign.
Thus, replace $A, C, 2$ with $-A, -C, -2$.

7.30. We are given

$$H(z) = H_c(s)\,\Big|_{s=\beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right]}$$

where α is a nonzero integer and β is a real number.

(a) *It is true for* $\beta > 0$.
Proof:

$$\begin{aligned} s &= \beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right] \\ s + sz^{-\alpha} &= \beta - \beta z^{-\alpha} \\ s - \beta &= -\beta z^{-\alpha} - sz^{-\alpha} \\ \beta - s &= z^{-\alpha}(\beta + s) \\ z^{-\alpha} &= \frac{\beta - s}{\beta + s} \\ z^{\alpha} &= \frac{\beta + s}{\beta - s} \end{aligned}$$

The poles s_k of a stable, causal, continuous-time filter satisfy the condition $\mathcal{R}e\{s\} < 0$. We want these poles to map to the points z_k in the z-plane such that $|z_k| < 1$. With $\alpha > 0$ it is also true that if $|z_k| < 1$ then $|z_k^{\alpha}| < 1$. Letting $s_k = \sigma + j\omega$ we see that

$$\begin{aligned} |z_k| &< 1 \\ |z_k^{\alpha}| &< 1 \\ |\beta + \sigma + j\Omega| &< |\beta - \sigma - j\Omega| \\ (\beta + \sigma)^2 + \Omega^2 &< (\beta - \sigma)^2 + \Omega^2 \\ 2\sigma\beta &< -2\sigma\beta \end{aligned}$$

But since the continuous-time filter is stable we have $\mathcal{R}e\{s_k\} < 0$ or $\sigma < 0$. That leads to

$$-\beta < \beta$$

This can only be true if $\beta > 0$.

(b) *It is true for* $\beta < 0$. The proof is similar to the last proof except now we have $|z^{\alpha}| > 1$.

(c) We have

$$\begin{aligned} z^2 &= \left.\frac{1+s}{1-s}\right|_{s=j\Omega} \\ |z^2| &= 1 \\ |z| &= 1 \end{aligned}$$

Hence, the $j\Omega$ axis of the s-plane is mapped to the unit circle of z-plane.

(d) First, find the mapping between Ω and ω.

$$\begin{aligned} j\Omega &= \frac{1 - e^{-j2\omega}}{1 + e^{-j2\omega}} \\ &= \frac{e^{j\omega} - e^{-j\omega}}{e^{j\omega} + e^{-j\omega}} \\ \Omega &= \tan(\omega) \\ \omega &= \tan^{-1}(\Omega) \end{aligned}$$

Therefore,

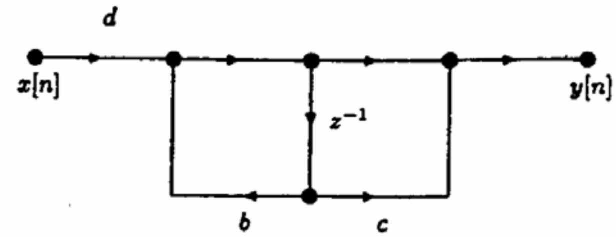
$$1 - \delta_1 \le |H(e^{j\omega})| \le 1 + \delta_1, \qquad \left\{|\omega| \le \frac{\pi}{4}\right\} \cup \left\{\frac{3\pi}{4} < |\omega| < \pi\right\}$$

Note that the highpass region $3\pi/4 \le |w| \le \pi$ is included because $\tan(\omega)$ is periodic with period π.

6.33.

$$H(z) = \frac{z^{-1} - 0.54}{1 - 0.54z^{-1}}.$$

(a)

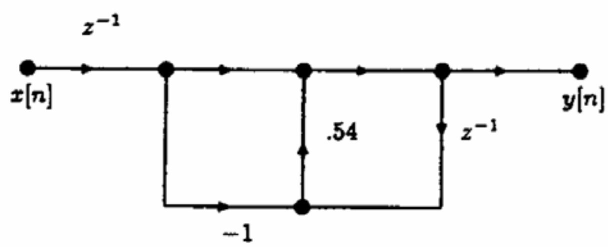


$$H(z) = \frac{cdz^{-1} + d}{1 - bz^{-1}}$$

so set $b = 0.54$, $c = -1.852$, and $d = -0.54$.

(b) With quantized coefficients $\hat{b}$, $\hat{c}$, and $\hat{d}$, $\hat{c}\hat{d} \neq 1$ and $\hat{d} \neq -\hat{b}$ in general, so the resulting system would not be allpass.

(c)

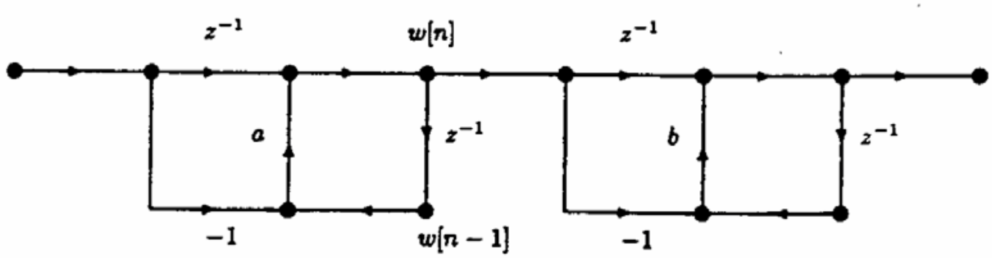


(d) Yes, since there is only one "0.54" to quantize.

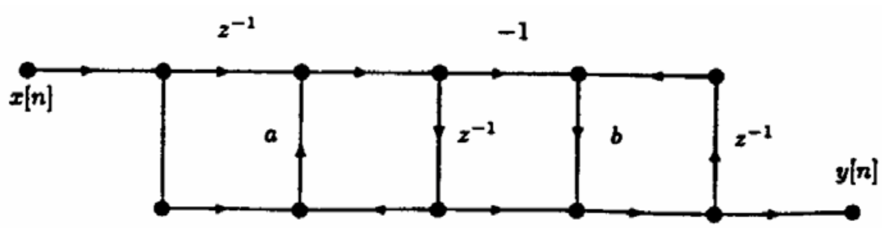
(e)

$$H(z) = \left(\frac{z^{-1} - a}{1 - az^{-1}}\right)\left(\frac{z^{-1} - b}{1 - bz^{-1}}\right)$$

Cascading two sections like (c) gives



The first delay in the second section has output $w[n-1]$ so we can combine with the second delay of the first section.



(f) Yes, same reason as part (d).