Digital Signal Processing

Midterm 1 Solution

Instructions

- Total time allowed for the exam is 80 minutes

- Some useful formulas:
  - Discrete Time Fourier Transform (DTFT)
    \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]
  
  - Inverse Fourier Transform
    \[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \]
  
  - Z Transform
    \[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]
1. (20 points) Match each of the pole-zero plots to the corresponding magnitude and phase plots given below. Write your answer in the space provided below each magnitude and phase plot.
2. (70 points) Consider a causal linear time-invariant system with impulse response $h[n]$. The $z$-transform of $h[n]$ is

$$H(z) = \frac{1 - z^{-1}}{1 - 0.5z^{-1}}$$

Consider the cascade configuration given in the figure below. Here we assume that $\alpha$ is a real number.

a. (10 pts) Draw the pole-zero plot of $H(z)$. Also, sketch the region of convergence. Is this system stable?

ROC: $\{z > \frac{1}{2}\}$. Since the unit circle lies in the region of convergence, the system is stable.

b. (10 pts) Assume that $\alpha = \frac{1}{3}$ and $x[n] = \left(\frac{1}{4}\right)^n u[n]$. Compute the output $y[n]$ for the above choice of input $x[n]$.

From the system diagram, we have $v[n] = \alpha^{-n} x[n]$, $w[n] = v[n] * h[n]$ and $y[n] = w[n] \alpha^n$. This implies

$$y[n] = \left(v[n] * h[n]\right) \alpha^n$$

$$= \left(\sum_{k=-\infty}^{\infty} v[n-k] h[k]\right) \alpha^n$$
\[
= \left( \sum_{k=-\infty}^{\infty} x[n-k] \alpha^{-(n-k)} h[k] \right) \alpha^n
\]
\[
= \left( \sum_{k=-\infty}^{\infty} x[n-k] \alpha^k h[k] \right)
\]
\[
= x[n] \ast (\alpha^n h[n])
\]

Taking the z-transform, we get
\[
Y(z) = X(z)H(\alpha^{-1}z)
\]
\[
= \frac{1}{1 - \frac{1}{4} z^{-1}} \cdot \frac{1 - \alpha z^{-1}}{1 - 0.5 \alpha z^{-1}}
\]
\[
= \frac{1}{1 - \frac{1}{4} z^{-1}} \cdot \frac{1 - \frac{1}{3} z^{-1}}{1 - \frac{1}{6} z^{-1}}
\]
\[
= \frac{-1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{6} z^{-1}}
\]

(1)

ROC of \(Y(z)\) is the intersection of the ROC of \(X(z)\) and \(H(z)\), which is \(|z| > \frac{1}{4}\). Now, taking the inverse z-transform of (1) we have

\[
y[n] = - \left( \frac{1}{4} \right)^n u[n] + 2 \left( \frac{1}{6} \right)^n u[n]
\]

c. (10 pts) Is the overall system enclosed by the dashed box linear? Explain your reasoning.

We have already shown that the output \(y[n] = x[n] \ast (\alpha^n h[n])\). Since the input and output are related via a convolution relationship, the system is a linear and time-invariant. Moreover, the impulse response of the overall system enclosed by the dashed box is \(g[n] = \alpha^n h[n]\).

d. (10 pts) Is the overall system enclosed by the dashed box time-invariant? Explain your reasoning.

See solution for part (c).

e. (10 pts) Find the response \(g[n]\) of the overall system within the enclosed box to the input \(x[n] = \delta[n]\).

Again from part (c), we have \(g[n] = \alpha^n h[n]\).

f. (10 pts) Plot the pole-zero plot of \(G(z)\), the z-transform of \(g[n]\). Also, sketch the region of convergence.
We know that $g[n] = \alpha^n h[n]$. This implies

$$G(z) = H\left(\alpha^{-1}z\right) = \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{6}z^{-1}}, \quad \text{ROC} : \{ |z| > \frac{1}{6} \}$$

The pole-zero plot of $G(z)$ is shown in the figure below.

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**g. (10 pts)** What happens to the pole-zero plot in part (f) if $\alpha = e^{-j\omega_0}$, for some real number $\omega_0$.

If $\alpha = e^{-j\omega_0}$, then the poles and zeros of $G(z)$ are obtained by rotating the corresponding poles and zeros of $H(z)$ by $\omega_0$ degrees in the clockwise direction. The region of convergence is $\{ |z| > 0.5 \}$. 

3. (40 points) A real-valued signal \( x(t) \) is known to be band-limited, i.e., \( X(j\omega) = 0 \), for \( |\omega| > W \). \( x(t) \) is first mixed by multiplying by \( \cos(\omega_0 t) \). The resulting signal \( v(t) \) is sampled uniformly (sampling time \( T \)) using the architecture given in the figure below. The periodic impulse train used for sampling is

\[
s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

Assume that \( x(t) = \frac{\sin W t}{W t} \) and \( \omega_0 = 2W \).

a. (10 pts) Compute and draw \( X(j\omega) \), the Fourier transform of \( x(t) \).

\[
X(j\omega) = \begin{cases} 
\frac{\pi}{W}, & -W \leq \omega \leq W \\
0, & \text{otherwise}
\end{cases}
\]

b. (10 pts) Compute and draw \( V(j\omega) \), the Fourier transform of \( v(t) \).

\[
v(t) = x(t) \cos(\omega_0 t)
\]

\[
\Rightarrow V(j\omega) = \frac{1}{2} [X(j(\omega + \omega_0)) + X(j(\omega - \omega_0))]
\]
c. (10 pts) Compute and draw $V_s(j\omega)$, the Fourier transform of $v_s(t)$.

$$v_s(t) = v(t)s(t)$$

$$\Rightarrow V_s(j\omega) = \frac{1}{2\pi} [V(j\omega) * S(j\omega)]$$

$$= \frac{1}{T} \left[ \sum_{k=-\infty}^{\infty} V(j(\omega - \Omega k)) \right]$$

where $\Omega = \frac{2\pi}{T}$. The Fourier transform $V_s(j\omega)$ is shown in Fig. 1.

(d) What is the minimum sampling rate $\Omega = \frac{2\pi}{T}$ required to reconstruct the original signal $x(t)$ from the discrete time signal $v_s[n]$.

The maximum frequency component of $v(t)$ is $3W$. Hence, from the Nyquist sampling theorem it is obvious that we can reconstruct $x(t)$ for sampling rates $\Omega > 6W$. Again from Nyquist sampling theorem, it is clear that we cannot recover $x(t)$ at a sampling rate $\Omega < 2W$. Now, the only question is whether we can recover $x(t)$ for sampling rates $2W \leq \Omega \leq 6W$. From Fig. 1 it is clear that if $\Omega = 2W$, the adjacent copies of $X(j\omega)$ don’t overlap. In this case we can reconstruct $x(t)$ by passing $v_s[n]$ through an ideal low-pass filter.

For $\Omega > 2W$, it is possible to reconstruct $x(t)$ from the samples $v_s[n]$, inspite of the overlap of the adjacent copies of $X(j\omega)$. The reconstruction process in this case is more than just low-pass filtering.
Figure 1: The Fourier transform $V_s(j\omega)$. The figures corresponds to the case when the sampling frequency $\Omega = 3.5W$. The different colors represent the copies of $V(j\omega)$ repeated periodically due to sampling.