EE 123
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Digital Signal Processing

Midterm 3

Name: ______________________ **SID:** ______________________

Instructions

- Total time allowed for the exam is 80 minutes
- Some useful formulas:
  - Discrete Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - Inverse Fourier Transform (IDTFT)

$$x[n] = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

  - Z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

1. (*45 points*) Decide whether the statements below are true or false. If true, give a proof. If false give a counterexample.

a. (*15 pts*) The bilinear transform method of filter design gives a uniquely invertible mapping between rational continuous transfer functions $H_c(s)$ and discrete-time rational transfer functions $H_d(z)$ once T_d has been specified.
i.e. Given that $H_d(z)$ was obtained from some particular $H_c(s)$ using a particular T_d, it could not have been obtained from any other $H'_c(s)$ using that same T_d.

$\boxed{\text{TRUE}}$

The bilinear transform is defined as

$$s \longmapsto \frac{2}{T_d}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \qquad -(1)$$

$$H_c(s) \longmapsto H_d(z) = H_c\left(\frac{2}{T_d}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right) \qquad -(2)$$

mapping in (1) is a one-one onto function from $\to \mathbb{C}$ and hence the bilinear transform is uniquely invertible.

b. (*15 pts*) The impulse-invariance method of filter design gives a uniquely invertible mapping between rational continuous transfer functions $H_c(s)$ and discrete-time rational transfer functions $H_d(z)$ once T_d has been specified.

$\boxed{\text{False}}$

Consider the following continuous time impulse responses $h_1(t)$ and $h_2(t)$

$$h_1(t) = \cos\left(\frac{2\pi t}{T_d}\right)$$

$$h_2(t) = \cos\left(\frac{4\pi t}{T_d}\right)$$, clearly $h_1(t) \neq h_2(t)$

Using the impulse-invariance method

$$h_1[n] = T_d\, h_1(nT_d)$$
$$= T_d \cos(2\pi n) = T_d$$

$$h_2[n] = T_d\, h_2(nT_d)$$
$$= T_d \cos(4\pi n) = T_d$$

$$\Rightarrow \; h_1[n] = h_2[n]$$

Hence impulse-invariance method is not uniquely invertible

c. (*15 pts*) The windowing method of FIR filter design gives a uniquely invertible mapping between rational continuous transfer functions $H_c(s)$ and discrete-time rational transfer functions $H_d(z)$ once the window-function W has been specified.

The following part needs to be added to the question

$$h[n] = h_c(nT), \quad h_d[n] = h[n]\,W[n]$$

The answer is $\boxed{\text{FALSE}}$, as neither of the above steps are uniquely invertible.

<u>Counter example</u>

$$h_1(t) = \cos\left(\frac{2\pi t}{T}\right)$$

$$h_2(t) = \cos\left(\frac{4\pi t}{T}\right)$$

For these, $h_1(nT) = h_2(nT)$, but $h_1(t) \neq h_2(t)$.

2. (*30 points*) We wish to design an FIR filter to approximate a desired $H_d(e^{j\omega})$ response that corresponds to a real impulse response and is furthermore both real and strictly positive at all frequencies ω.
You have chosen to implement the FIR filter using an 11 point triangular (Bartlett) window $w[n]$. It turns out that $h[n] = h_d[n]w[n]$ has acceptable characteristics in frequency domain. ($h_d[n]$ is obtained from the IDTFT of H_d)
Show how to calculate coefficients for both a direct form implementation as well as an implementation as a cascade of two 6 point FIR filters.

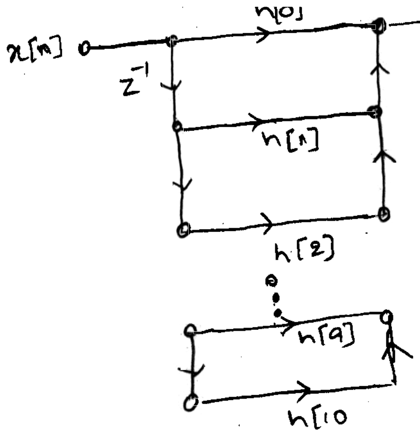
Using the Bartlett window we can compute

$$h[n] = h_d[n]\, w[n] \qquad n = 0, \quad 10$$

Then we can compute the transfer function

$$H(z) = \sum_{n=0}^{10} h[n]\, z^{-n}$$

The direct form implementation of this system is given by



Also, from the fundamental theorem of Algebra, we can write

$$H(z) = A(z)\, B(z)$$ where both $A(z), B(z)$ are degree 5 polynomials.

Hence, we can implement $H(z)$ as a cascade of two 6 point FIR filters. This can be done by cascading the direct form implementations of $A(z)$ and $B(z)$

3. (*90 points*) Consider a stable causal continuous-time system with rational transfer function:

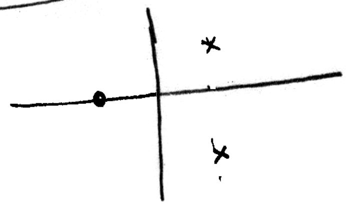
$$H_c(s) = \frac{s+c}{(s+a)^2+b}$$

a. (*10 pts*) Where are the poles and zeros of this continuous-time system as a function of the real numbers a, b, c. Sketch the various cases that can occur and label which cases are compatible with the stable causal assumption.

$$H_c(s) = \frac{s+c}{(s+a+\sqrt{-b})(s+a-\sqrt{-b})} \Rightarrow \text{poles at } -a \pm \sqrt{-b}$$

zeros at $-c$

Case I: $a \leq 0$, $b \geq 0$



$\boxed{\text{Not Compatible}}$

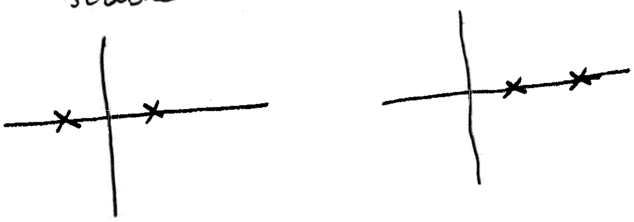
The location of 'c' does not matter for stability and causality

Case II $a < 0$, $b < 0$

$\max\{-a+\sqrt{-b} \quad -a-\sqrt{-b}\} > 0$

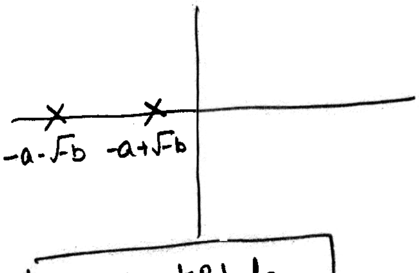
$\Rightarrow$ at least one pole in the right half plane

$\Rightarrow$ $\boxed{\text{Not Compatible}}$ with the stable - causal assumption



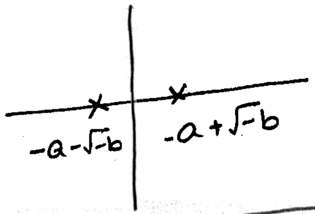
Case III $a \geq 0$, $b < 0$

$-a+\sqrt{-b} < 0$



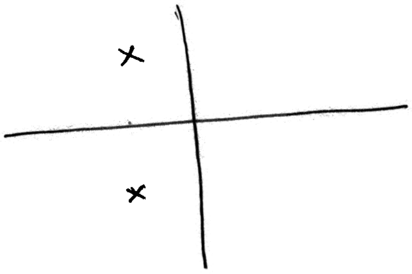
$\boxed{\text{Compatible}}$

$-a+\sqrt{b} \geq 0$



$\boxed{\text{Not Compatible}}$

Case IV: $a > 0$, $b \geq 0$



$\boxed{\text{Compatible}}$

b. (*15 pts*) Suppose now that $b < 0$. Use impulse-invariance to determine a discrete time system $H_i(z)$ such that $h_i[n] = h_c(nT)$.

$b<0$, poles $= -a-\sqrt{-b},\ -a+\sqrt{-b}$

Let $\alpha_1 \triangleq -a-\sqrt{-b}$, $\alpha_2 \triangleq -a+\sqrt{-b}$

$$\Rightarrow H_c(s) = \frac{(s+c)}{(s-\alpha_1)(s-\alpha_2)} = \frac{A}{s-\alpha_1} + \frac{B}{s-\alpha_2}$$

By solving for A, B, we get

$$A = \frac{\alpha_1+c}{\alpha_1-\alpha_2}, \quad B = \frac{\alpha_2+c}{\alpha_2-\alpha_1}$$

Hence, the discrete time system is given by

$$H_i(z) = \frac{A}{1-e^{\alpha_1 T}z^{-1}} + \frac{B}{1-e^{\alpha_2 T}z^{-1}}$$

c. (*40 pts*) Draw how to implement the system $H_i(z)$ using Direct Form 1, Direct Form 2, Cascade Form, and Parallel Form.
Comment on the quantitative impact of rounding noise in each case.

$H_i(z)$ can be re-written as

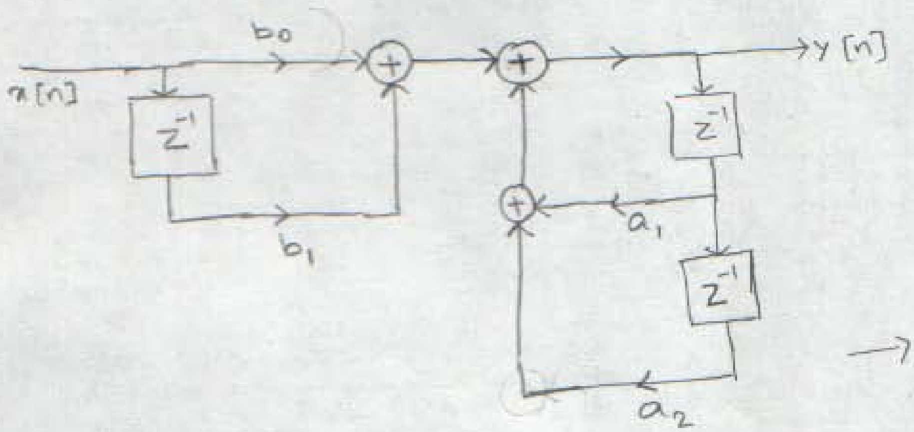
$$H_i(z) = \frac{(A+B) - (Ae^{\alpha_2 T} + Be^{\alpha_1 T})z^{-1}}{1 - (e^{\alpha_1 T} + e^{\alpha_2 T})z^{-1} + e^{(\alpha_1+\alpha_2)T} z^{-2}} \quad - (*)$$

This can be written as the following difference equation

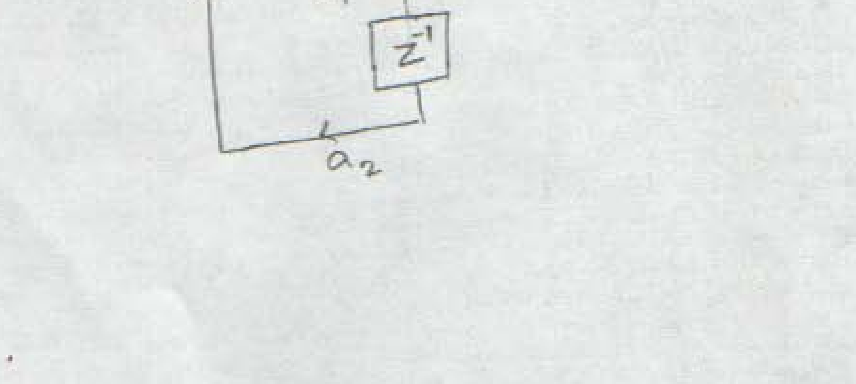
$$y_i[n] = a_1 y_i[n-1] + a_2 y_i[n-2] + b_0 x[n] + b_1 x[n-1]$$

where the coefficients can be read from $(*)\rightarrow$

$$\boxed{\begin{aligned} b_0 &= A+B \\ b_1 &= (Ae^{\alpha_2 T} + Be^{\alpha_1 T}) \\ a_1 &= (e^{\alpha_1 T} + e^{\alpha_2 T}) \\ a_2 &= -e^{(\alpha_1+\alpha_2)T} \end{aligned}}$$



$\rightarrow$ Direct Form I

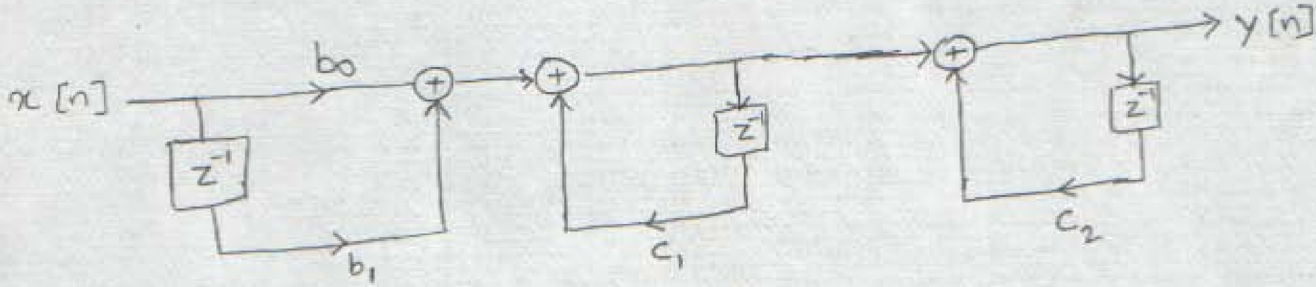


Direct Form II

For cascade form we can write $H_i(z)$ as

$$H_i(z) = \left(\frac{b_0 + b_1 z^{-1}}{1 - e^{\alpha_1 T} z^{-1}}\right)\left(\frac{1}{1 - e^{\alpha_2 T} z^{-1}}\right)$$

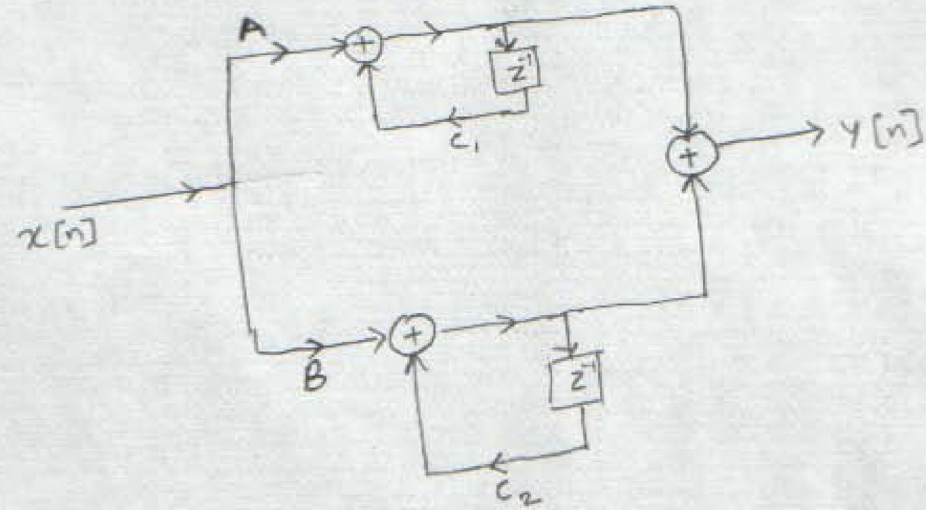
$$= \left(\frac{b_0 + b_1 z^{-1}}{1 - c_1 z^{-1}}\right)\left(\frac{1}{1 - c_2 z^{-1}}\right), \quad \text{where } \begin{aligned} c_1 &= e^{\alpha_1 T} \\ c_2 &= e^{\alpha_2 T} \end{aligned}$$



Cascade Form

For parallel form, we have

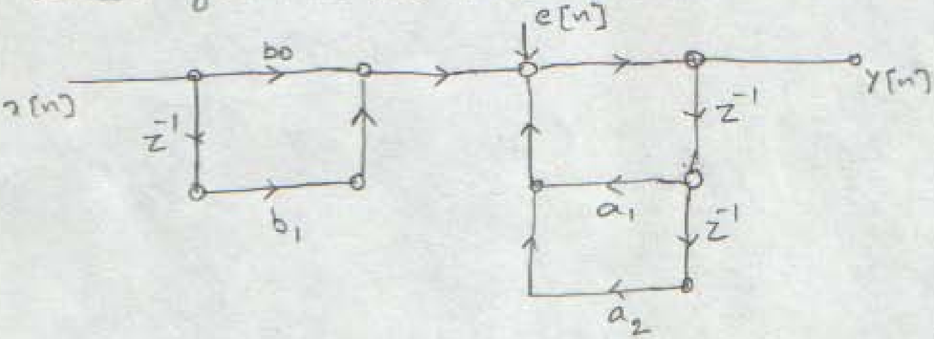
$$H_i(z) = \frac{A}{1 - c_1 z^{-1}} + \frac{B}{1 - c_2 z^{-1}}$$



Parallel Form

Impact of rounding noise

1) In direct form I, the linear-noise model is as follows.

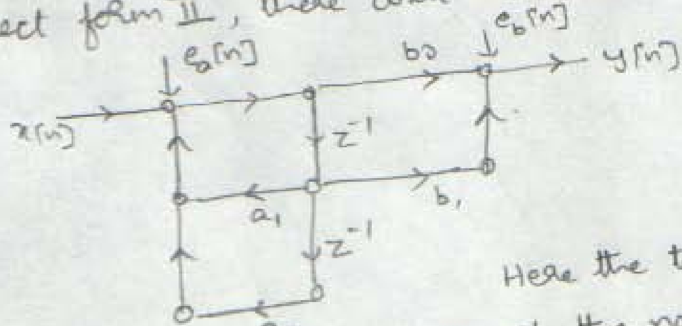


where the additive noise e[n] has zero mean and variance

$$\sigma_e^2 = 4\,\frac{2^{-2B}}{12}$$ (where 'B+1' pt arithematic is used)

The noise passes only through the poles part of the system.

2) In direct form II, there will be two noise sources.

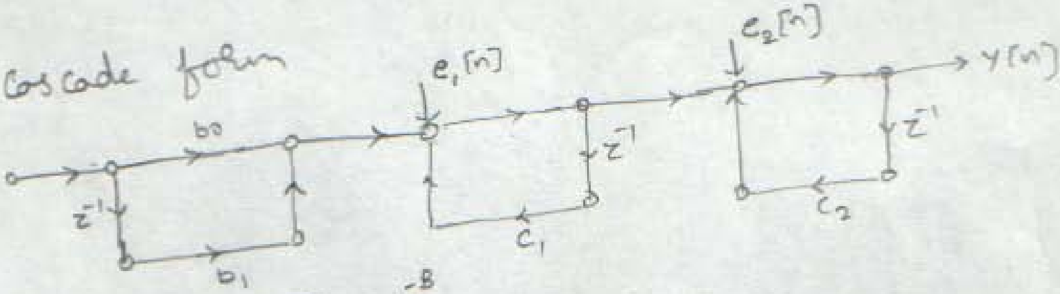


$$\sigma_{e_a}^2 = 2\,\frac{2^{-B}}{12}$$

$$\sigma_{e_b}^2 = \frac{2^{-B}}{12}$$

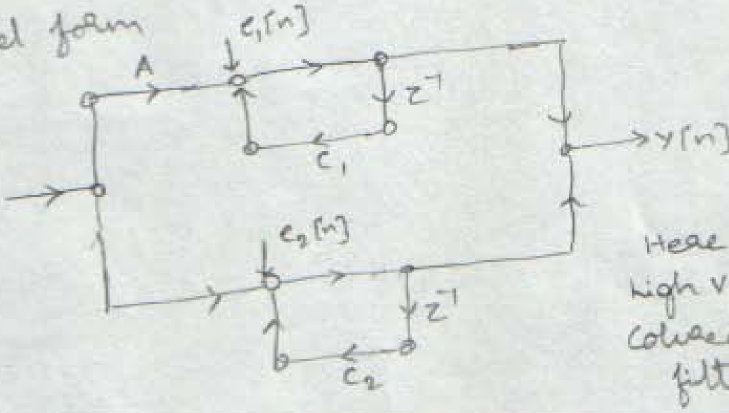
Here the total noise variance is lower, but one of the noise term, $e_a[n]$ is passed through the whole system, both poles & filters.

3) In cascade form



$$\sigma_{e_1}^2 = 2\,\frac{2^{-B}}{12}, \quad \sigma_{e_2}^2 = \frac{2^{-B}}{12}$$

4) In parallel form



$$\sigma_{e_1}^2 = 2\cdot\frac{2^{-B}}{12}$$

$$\sigma_{e_2}^2 = 2\cdot\frac{2^{-B}}{12}$$

Here, both the noise terms have high variance, but they are colored only by single pole filters.

d. (*15 pts*) Continue to assume $b < 0$. Figure out a discrete-time system $H_s(z)$ such that

$$\sum_{k=-\infty}^{n} h_s[k] = \int_{-\infty}^{nT} h_c(\tau)d\tau$$

Is it rational? (Hint: this corresponds to invariance under a step-input)

Let $S_c(t) \triangleq \int_{-\infty}^{t} h_c(\tau)\,d\tau$, $S_2[n] \triangleq \sum_{k=-\infty}^{\infty} h_s[k]$

Then, we must have $S_2[n] = S_c(nT)$, which is just the impulse invariance realationship wetween $S_c(z)$ and $S_2(z)$.

Now, $$S_c(s) = \frac{H_c(s)}{s} = \frac{(s+c)}{s(s-\alpha_1)(s-\alpha_2)}$$

$$= \frac{A_1}{s} + \frac{A_2}{s-\alpha_1} + \frac{A_3}{s-\alpha_2}$$

Solving for A_1, A_2, A_3, we get

$$A_1 = \frac{c}{\alpha_1\alpha_2}, \quad A_2 = \frac{\alpha_1 + c}{\alpha_1(\alpha_1 - \alpha_2)}, \quad A_3 = \frac{\alpha_2 + c}{\alpha_2(\alpha_2 - \alpha_1)}$$

Hence,

$$S_2(z) = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 - e^{\alpha_1 T} z^{-1}} + \frac{A_3}{1 - e^{\alpha_2 T} z^{-1}}$$

We also know that

$$S_2(z) = \frac{H_s(z)}{1 - z^{-1}}$$

$$\Rightarrow H_s(z) = S_2(z)(1 - z^{-1})$$

$$= A_1 + \frac{A_2(1 - z^{-1})}{1 - e^{\alpha_1 T} z^{-1}} + \frac{A_3(1 - z^{-1})}{1 - e^{\alpha_2 T} z^{-1}}$$

e. *(10 pts)* Is $H_i(z) = H_s(z)$ always, sometimes, or never?

From part (b), we have

$$H_i(z) = \frac{A}{1-e^{\alpha_1 T}z^{-1}} + \frac{B}{1-e^{\alpha_2 T}z^{-1}} = \frac{(A+B)-(Ae^{\alpha_2 T}+Be^{\alpha_1 T})z^{-1}}{(1-e^{\alpha_1 T}z^{-1})(1-e^{\alpha_2 T}z^{-1})}$$

and from part (d)

$$H_s(z) = A_1 + \frac{A_2(1-z^{-1})}{1-e^{\alpha_1 T}z^{-1}} + \frac{A_3(1-z^{-1})}{1-e^{\alpha_2 T}z^{-1}}$$

$$= \frac{A_1(1-e^{\alpha_1 T}z^{-1})(1-e^{\alpha_2 T}z^{-1}) + A_2(1-z^{-1})(1-e^{\alpha_2 T}z^{-1}) + A_3(1-z^{-1})(1-e^{\alpha_1 T}z^{-1})}{(1-e^{\alpha_1 T}z^{-1})(1-e^{\alpha_2 T}z^{-1})}$$

Therefore, for $H_i(z) = H_s(z)$, we must have

$$A_1 + A_2 + A_3 = A + B \quad -(1)$$

$$(A_1+A_3)e^{\alpha_1 T} + (A_1+A_2)e^{\alpha_2 T} + (A_2+A_3) = Ae^{\alpha_2 T} + Be^{\alpha_1 T} \quad -(2)$$

$$A_1 e^{(\alpha_1+\alpha_2)T} + A_2 e^{\alpha_2 T} + A_3 e^{\alpha_1 T} = 0 \quad -(3)$$

Therefore, $H_i(z) = H_s(z)$ sometimes.