Digital Signal Processing

Midterm 3

Name: ____________________________  SID: ____________________________

Instructions

- Total time allowed for the exam is 80 minutes
- Some useful formulas:
  - Discrete Time Fourier Transform (DTFT)
    \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]
  - Inverse Fourier Transform (IDTFT)
    \[ x[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(e^{j\omega})e^{j\omega n} d\omega \]
  - Z Transform
    \[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]
1. (45 points) Decide whether the statements below are true or false. If true, give a proof. If false give a counterexample.

   a. (15 pts) The bilinear transform method of filter design gives a uniquely invertible mapping between rational continuous transfer functions $H_c(s)$ and discrete-time rational transfer functions $H_d(z)$ once $T_d$ has been specified.

   i.e. Given that $H_d(z)$ was obtained from some particular $H_c(s)$ using a particular $T_d$, it could not have been obtained from any other $H'_c(s)$ using that same $T_d$.

   **TRUE**

   The bilinear transform is defined as

   $$ s \longrightarrow \frac{2}{T_d} \left[ \frac{1 - z^{-1}}{1 + z^{-1}} \right] $$

   (1)

   $$ H_c(s) \longrightarrow H_d(z) = H_c \left( \frac{2}{T_d} \left[ \frac{1 - z^{-1}}{1 + z^{-1}} \right] \right) $$

   (2)

   Mapping in (1) is a one-one onto function from $\mathbb{C}$ to $\mathbb{C}$ and hence the bilinear transform is uniquely invertible.
b. (15 pts) The impulse-invariance method of filter design gives a uniquely invertible mapping between rational continuous transfer functions $H_c(s)$ and discrete-time rational transfer functions $H_d(z)$ once $T_d$ has been specified.

**False**

Consider the following continuous time impulse responses $h_1(t)$ and $h_2(t)$

$$h_1(t) = \cos\left(\frac{2\pi t}{T_d}\right)$$

$$h_2(t) = \cos\left(\frac{4\pi t}{T_d}\right), \text{ clearly } h_1(t) \neq h_2(t)$$

Using the impulse-invariance method

$$h_1[n] = T_d \cdot h_1(nT_d) = T_d \cos(2\pi n) = T_d$$

$$h_2[n] = T_d \cdot h_2(nT_d) = T_d \cos(4\pi n) = T_d$$

$$\Rightarrow h_1[n] = h_2[n]$$

Hence impulse-invariance method is not uniquely invertible.
c. (15 pts) The windowing method of FIR filter design gives a uniquely invertible mapping between rational continuous transfer functions \( H_c(s) \) and discrete-time rational transfer functions \( H_d(z) \) once the window-function \( W \) has been specified.

The following part needs to be added to the question:

\[
h[n] = h_c(nT), \quad h_d[n] = h[n] W[n]
\]

The answer is **FALSE**, as neither of the above steps are uniquely invertible.

Counterexample:

\[
h_1(t) = \cos \left( \frac{2\pi T}{T} \right)
\]

\[
h_2(t) = \cos \left( \frac{4\pi T}{T} \right)
\]

For these, \( h_1(nT) = h_2(nT) \), but \( h_1(t) \neq h_2(t) \).
2. (30 points) We wish to design an FIR filter to approximate a desired $H_d(e^{j\omega})$ response that corresponds to a real impulse response and is furthermore both real and strictly positive at all frequencies $\omega$.
You have chosen to implement the FIR filter using an 11 point triangular (Bartlett) window $w[n]$. It turns out that $h[n] = h_d[n]w[n]$ has acceptable characteristics in frequency domain. ($h_d[n]$ is obtained from the IDTFT of $H_d$)
Show how to calculate coefficients for both a direct form implementation as well as an implementation as a cascade of two 6 point FIR filters.

Using the Bartlett window we can compute

$$h[n] = h_d[n]w[n] \quad n = 0, \ldots, 10$$

Then we can compute the transfer function

$$H(z) = \sum_{n=0}^{10} h[n] z^{-n}$$

The direct form implementation of this system is given by:

![Direct Form FIR Filter Diagram]

Also, from the fundamental theorem of Algebra, we can write

$$H(z) = A(z)B(z)$$

where both $A(z), B(z)$ are degree 5 polynomials.

Hence, we can implement $H(z)$ as a cascade of two 6 point FIR filters. This can be done by cascading the direct form implementations of $A(z)$ and $B(z)$.
3. (90 points) Consider a stable causal continuous-time system with rational transfer function:

\[ H_c(s) = \frac{s + c}{(s + a)^2 + b} \]

a. (10 pts) Where are the poles and zeros of this continuous-time system as a function of the real numbers \( a, b, c \). Sketch the various cases that can occur and label which cases are compatible with the stable causal assumption.

\[ H_c(s) = \frac{s + c}{(s + a + \sqrt{b})(s + a - \sqrt{b})} \implies \text{poles at } -a \pm \sqrt{b} \]
\[ \text{zeros at } -c \]

**Case I:** \( a < 0, \ b > 0 \)
- Not Compatible

The location of 'c' does not matter for stability and causality

**Case II:** \( a < 0, \ b < 0 \)
max \( \{-a + \sqrt{b}, -a - \sqrt{b}\} > 0 \)
- Not Compatible with the stable-casual assumption

**Case III:** \( a > 0, \ b < 0 \)
- Not Compatible

**Case IV:** \( a > 0, \ b > 0 \)
- Compatible
b. (15 pts) Suppose now that $b < 0$. Use impulse-invariance to determine a discrete time system $H_1(z)$ such that $h_k[n] = h_c(nT)$.

$b < 0$, poles $= -a - \sqrt{b}, \ -a + \sqrt{b}$

Let $d_1 = -a - \sqrt{b}$, $d_2 = -a + \sqrt{b}$

$\Rightarrow H_c(s) = \frac{(s + c)}{(s - d_1)(s - d_2)} = \frac{A}{s - d_1} + \frac{B}{s - d_2}$

By solving for $A$, $B$, we get

$A = \frac{d_1 + c}{d_1 - d_2}$, $B = \frac{d_2 + c}{d_2 - d_1}$

Hence, the discrete time system is given by

$H_1(z) = \frac{A}{1 - e^{d_1 T} z^{-1}} + \frac{B}{1 - e^{d_2 T} z^{-1}}$
c. (40 pts) Draw how to implement the system $H_i(z)$ using Direct Form 1, Direct Form 2, Cascade Form, and Parallel Form. Comment on the quantitative impact of rounding noise in each case.

$H_i(z)$ can be re-written as

$$H_i(z) = \frac{(A+B) - (Ae^{i\theta_1 T} + Be^{i\theta_2 T})z^{-1}}{1 - (e^{i\theta_1 T} + e^{i\theta_2 T})z^{-1} + e^{i(x_1 + x_2)T}z^{-2}} \quad (*)$$

This can be written as the following difference equation:

$$y_2[n] = a_1 y_2[n-1] + a_2 y_2[n-2] + b_0 x[n] + b_1 x[n-1]$$

where the coefficients can be read from $(*)$:

- $b_0 = A + B$
- $b_1 = (Ae^{i\theta_1 T} + Be^{i\theta_2 T})$
- $a_1 = (e^{i\theta_1 T} + e^{i\theta_2 T})$
- $a_2 = -e^{i(x_1 + x_2)T}$

→ Direct Form I

→ Direct Form II
For cascade form we can write \( H_2(z) \) as

\[
H_2(z) = \left( \frac{b_0 + b_1 z^{-1}}{1 - c_1 z^{-1}} \right) \left( \frac{1}{1 - c_2 z^{-1}} \right)
\]

\[
= \left( \frac{b_0 + b_1 z^{-1}}{1 - c_1 z^{-1}} \right) \left( \frac{1}{1 - c_2 z^{-1}} \right), \quad \text{where} \quad c_1 = e^{\alpha_1 T}, \quad c_2 = e^{\alpha_2 T}
\]

Cascaded Form

For parallel form, we have

\[
H_2(z) = \frac{A}{1 - c_1 z^{-1}} + \frac{B}{1 - c_2 z^{-1}}
\]

Parallel Form
Impact of rounding noise

1) In direct form I, the linear noise model is as follows:

\[
\begin{align*}
&x[n] \rightarrow z^{-1} \rightarrow \frac{b_0}{1} \rightarrow z^{-1} \rightarrow \frac{a_1}{1} \rightarrow z^{-1} \rightarrow \frac{a_2}{1} \rightarrow y[n] \\
&\text{where the additive noise } e[n] \text{ has zero mean and variance } \\
&\sigma_e^2 = 4 \cdot \frac{2^{-8}}{12} \quad (\text{where } B=1 \text{ pt arithmetic is used})
\end{align*}
\]

The noise passes only through the poles part of the system.

2) In direct form II, there will be two noise sources:

\[
\begin{align*}
&x[n] \rightarrow z^{-1} \rightarrow \frac{b_0}{1} \rightarrow z^{-1} \rightarrow \frac{a_1}{1} \rightarrow \frac{a_2}{1} \rightarrow y[n] \\
&\frac{e_a[n]}{e_b[n]} \rightarrow \frac{e_b[n]}{1} \\
&\sigma_e^2 = 2 \cdot \frac{2^{-8}}{12} \\
&\sigma_{e_a}^2 = \frac{2^{-8}}{12} \\
&\sigma_{e_b}^2 = \frac{2^{-8}}{12}
\end{align*}
\]

Here the total noise variance is lower, but one of the noise term, \(e_a[n]\) is passed through the whole system, both poles as filters.

3) In cascade form:

\[
\begin{align*}
&x[n] \rightarrow z^{-1} \rightarrow \frac{b_0}{1} \rightarrow z^{-1} \rightarrow \frac{a_1}{1} \rightarrow z^{-1} \rightarrow \frac{a_2}{1} \rightarrow y[n] \\
&\sigma_e^2 = 2 \cdot \frac{2^{-8}}{12}, \quad \sigma_{e_2}^2 = \frac{2^{-8}}{12}
\end{align*}
\]

4) In parallel form:

\[
\begin{align*}
&x[n] \rightarrow z^{-1} \rightarrow \frac{b_0}{1} \rightarrow z^{-1} \rightarrow \frac{a_1}{1} \rightarrow z^{-1} \rightarrow \frac{a_2}{1} \rightarrow y[n] \\
&\sigma_e^2 = 2 \cdot \frac{2^{-8}}{12}, \quad \sigma_{e_2}^2 = \frac{2^{-8}}{12}
\end{align*}
\]

Here, both the noise terms have high variance, but they are caused only by single pole filters.
d. (15 pts) Continue to assume \( b < 0 \). Figure out a discrete-time system \( H_d(z) \) such that

\[
\sum_{k=-\infty}^{n} h_n[k] = \int_{-\infty}^{nT} h_c(r) dr
\]

Is it rational? (Hint: this corresponds to invariance under a step-input)

Let \( S_c(t) A = \int_{-\infty}^{t} h_c(z) dz , \ S_2[n] = \sum_{k=-\infty}^{n} h_n[k] \)

Then, we must have \( S_2[n] = S_c(nT) \), which is just the impulse
invariance relationship between \( S_c(z) \) and \( S_2(z) \).

Now,

\[
S_c(s) = \frac{H_c(s)}{s} = \frac{(s+c)}{s(s-d_1)(s-d_2)}
\]

\[
= \frac{A_1}{s} + \frac{A_2}{s-d_1} + \frac{A_3}{s-d_2}
\]

Solving for \( A_1, A_2, A_3 \), we get

\[
A_1 = \frac{c}{d_1 d_2}, \quad A_2 = \frac{d_1 + c}{d_1 (d_1 - d_2)}, \quad A_3 = \frac{d_2 + c}{d_2 (d_2 - d_1)}
\]

Hence,

\[
S_2(z) = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 - e^{d_1 T} z^{-1}} + \frac{A_3}{1 - e^{d_2 T} z^{-1}}
\]

We also know that

\[
S_2(z) = \frac{H_5(z)}{1 - z^{-1}}
\]

\[
\Rightarrow H_5(z) = S_2(z) (1 - z^{-1})
\]

\[
= A_1 + \frac{A_2 (1 - z^{-1})}{1 - e^{d_1 T} z^{-1}} + \frac{A_3 (1 - z^{-1})}{1 - e^{d_2 T} z^{-1}}
\]
(10 pts) Is $H_i(z) = H_o(z)$ always, sometimes, or never?

From part (b), we have

$$H_o(z) = \frac{A}{1-e^{\alpha_{1}T}z^{-1}} + \frac{B}{1-e^{\alpha_{2}T}z^{-1}} = \frac{(A+B)-(Ae^{\alpha_{1}T}+Be^{\alpha_{1}T})z^{-1}}{(1-e^{\alpha_{1}T}z^{-1})(1-e^{\alpha_{2}T}z^{-1})}$$

and from part (d)

$$H_o(z) = A_1 + \frac{A_2(1-z^{-1})}{1-e^{\alpha_{1}T}z^{-1}} + \frac{A_3(1-z^{-1})}{1-e^{\alpha_{2}T}z^{-1}}$$

$$= \frac{A_1(1-e^{\alpha_{1}T}z^{-1})(1-e^{\alpha_{2}T}z^{-1}) + A_2(1-z^{-1})(1-e^{\alpha_{2}T}z^{-1}) + A_3(1-z^{-1})(1-e^{\alpha_{1}T}z^{-1})}{(1-e^{\alpha_{1}T}z^{-1})(1-e^{\alpha_{2}T}z^{-1})}$$

Therefore, for $H_i(z) = H_o(z)$, we must have

$$A_1 + A_2 + A_3 = A + B \quad - (1)$$

$$(A_1+A_2)e^{\alpha_{1}T} + (A_1+A_2)e^{\alpha_{2}T} + (A_2+A_3) = Ae^{\alpha_{1}T} + Be^{\alpha_{1}T} \quad - (2)$$

$$A_1 e^{(\alpha_{1}+\alpha_{2})T} + A_2 e^{\alpha_{2}T} + A_3 e^{\alpha_{1}T} = 0 \quad - (3)$$

Therefore, $H_o(z) = H_o(z)$ sometimes.