EE 123: DSP **Spring 2007**

Lecture 1 — January 18

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Outline

1. Discrete time signals
2. Property of systems.
3. LTI system

1.1 Discrete-time Signals and Systems

1.1.1 Discrete-time signals: sequences

Discrete-time signals are represented mathematically as sequences of numbers. A sequence of numbers x , in which the nth number in the sequence denoted $x[n]$, is formally written as

$$x = \{x[n]\}, -\infty < n < \infty,$$

where n is an integer.

Discrete-time system is defined as a transformation or an operator that maps an input sequence $x[n]$ into an output sequence with values $y[n]$. This can be denoted as

$$y = T\{x[n]\}$$

1.1.2 Basic sequences and sequence operations

The unit sample sequence (often referred as a **discrete time impulse** or simply as an impulse)is defined as the sequence

$$\delta[n] = \{ \begin{array}{l} 1 \text{ if n=0} \\ 0 \text{ otherwise} \end{array}$$

Thus, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Note: the discrete-time impulse is different from the continuous time impulse or the dirac delta function, which takes the form of

$$\delta(t) = \{ \begin{matrix} > \infty \text{ if t=0} \\ 0 \text{ otherwise} \end{matrix}$$

The dirac delta function has the property of $\int_{-\infty}^{\infty} \delta(t)dt = 1$

The **unit step sequence** is given by

$$u[n] = \{ \begin{matrix} 1 \text{ n} \geq 0 \\ 0 \text{ n} < 0 \end{matrix}$$

The impulse sequence can also be expressed as:

$$\delta[n] = u[n] - u[n-1]$$

The **exponential sequences** are very important in representing and analyzing linear time-invariant discrete time systems. The complex exponential sequence has the general form of

$$x[n] = A\alpha^n = |A|\, e^{j\phi}\, |\alpha|^n\, e^{jw_0 n} = |A|\, |\alpha|\, [cos(w_0 n + \phi) + jsin(w_0 n + \phi)]$$

As shown above, the complex exponential can be expressed in the sinusoidal form.

An important difference between continuous time and discrete time complex exponential and sinusoids concerns their periodicity. In the discrete time case, $x[n]$ is periodic with period N if $x[n+N] = x[n]$, for all n. For instance $sin(n)$ and most other discrete time sinusoids are NOT periodic whereas $\sin(\pi n/2)$ is periodic. In the continuous time domain, a sinusoidal signal is always periodic, with the period equal to 2π divided by the frequency.

A **conjugate symmeteric sequence** $x_e[n]$ is defined as a sequence for which $x_e[n] = x_e^*[-n]$, and a **conjugate anti-symmetric sequence** $x_o[n]$ is defined as a sequence for which $x_o[n] = -x_o^*[-n]$, where * denotes complex conjugation. Any sequence $x[n]$ can be expressed as a sum of conjugate symmetric and conjugate anti-symmetric sequence:

$$x[n] = x_e[n] + x_o[n] \text{ with } x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) \text{ and } x_o[n] = \frac{1}{2}(x[n] - x^*[-n])$$

1.2 Property of Systems

1.2.1 Memoryless Systems

A system is refered to as **memoryless** if the output $y[n]$ depends only on n and $x[n]$.

1.2.2 Causality

A system is **casual** if $y[n]$ only depends on n and $x[k]$ for $k \leq n$. Consequently, a system is **anticausal** if $y[n]$ depends on n and $x[k]$ for $k \geq n$. For a strictly causual or anticausual system, $k = n$ is excluded from the set.

Note: if a system is memoryless, then it is causal as well.

1.2.3 Time invariant systems

A **time invariant system** is a system for which a time shift or delay of the input sequence causes a corresponding shift in the output. Specifically, the system, $y = T\{x[n]\}$, is said to be time invariant if $y[n-N] = T(D_N[x])$ or $D_N(y) = T(D_N[x])$ or $D_N(T(x)) = T(D_N(x))$ for all N. Time invariance can also be referred as commutability with delays.

1.2.4 Linear systems

If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is **linear** if and only if

$$T\{\alpha x_1[n] + \beta x_2[n]\} = \alpha T\{x_1[n]\} + \beta T\{x_2[n]\}$$

for arbitrary constants α and β.

1.2.5 Stability

A system is **stable** in the **bounded-input, bounded-output (BIBO)** sense if and only if for every bounded input signal x, output signal y is bounded. An example of an *unstable system* is an accumulator system defined by $y[n] = \sum_{k=-\infty}^{n} x[k]$. Note, memoryless doesn't imply stability; such example would be $y[n] = n \cdot x[n]$.

1.2.6 Summary

Given the five properties of a system (memoryless, causal, time invariant, linear, stability) and memoryless implying causal, there are 24 possible classes of systems satisfying or not satisfying a set of the above properties. It would be a good exercise to identify 24 such systems for each case.

1.3 Linear time-invariant (LTI) systems

1.3.1 Definitions

A particularly important class of systems consists of those that are linear and time invariant. We know that the input sequence $x[n]$ can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

And the output $y[n]$ can be expressed as

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}.$$

Let $h_k[n]$ be the response of the system to $\delta[n-k]$. Since T is linear, we can write

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\left\{\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

Given the system is time invariant, the response to $\delta[n-k]$ is $h[n-k]$. We can rewrite the output as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

which is commonly known as the **convolution sum**. We say that $y[n]$ is the convolution of $x[n]$ with $h[n]$ and can represent this by the notation

$$y[n] = x[n] * h[n].$$

As a consequence, a linear time invariant system is completely characterized by its impulse response $h[n]$ in the sense that, given $h[n]$, it is possible to compute the output $y[n]$ due to any input $x[n]$. For example, the accumulator system has $h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$. Computing the convolution sum can be described as a drag, multiply and add process. The sequence $h[n-k]$ can be obtained through the following two steps: 1. reflecting $h[k]$ about the origin to obtain $h[-k]$; 2. shifting the origin of the reflected sequence to $k = n$.

1.3.2 Examples of LTI Systems

An example of a memoryless LTI system can have an impulse response of: $h[n] = \alpha\delta[n]$.

A n example of a causal LTI system is

$$h[n] = \{ \begin{array}{l} \text{any values, for n} \geq 0 \\ 0, n \leq 0 \end{array}$$

A causal stable LTI can have an impulse response of

$$h[n] = \{ \begin{array}{l} \text{absolute summable, for instance, } \sum |h[n]| \text{must converge for } n \geq 0 \\ 0, n < 0 \end{array}$$

1.3.3 Linear constant-coefficient difference equations

An important subclass of linear time-invariant systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an Nth-order linear constant-coefficient difference equation(LCCDE) of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

For example, the accumulator system mentioned in earlier section $y[n] = \sum_{k=-\infty}^{n} x[k]$ can be rewritten in the LCCDE form. We can first write the output for $n-1$ as

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k].$$

Now,

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k].$$

Through substitution,

$$y[n] = x[n] + y[n-1].$$

And,

$$y[n] - y[n-1] = x[n]$$

Thus, the input and output satisfy a linear constant-coefficient difference equation with $N = 1, a_0 = 1, a_1 = -1, M = 0$ amd $b_0 = 1$.

LCCDE systems can be implemented using a finite number of registers.