EE 123: Digital Signal Processing **Spring 2007**

Lecture 4 — January 25

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4.1 Outline

These notes cover the following topics:

- Sampling and Interpolation (continued)
- DTFT properties review
- Z-transform review

4.2 Sampling and Interpolation

In the previous lecture, we saw that it is possible to exactly reconstruct a signal from its samples if the original signal is bandlimited and if we take samples frequently enough. In addition, we saw that if these conditions does not hold, then aliasing occurs and we can not recover our original signal.

In this notes we will focus on the processes of sampling and interpolation. Our goal is to understand better the effects of taking samples of a continuous signal and the effects of the "ideal" reconstruction.

4.2.1 Interpolator

An interpolator is a discrete-to-continuous-time (D/C) converter that reconstructs a continuos signal from its samples. For convenience, we want our "ideal" interpolator to satisfy the following two properties:

1. return the "right answer" for complex exponentials
2. be an LTI system

The motivation for the first property is that many practical signals are complex exponentials or can be decomposed as a linear combination of them (using the Fourier transform). The motivation for the second property is that LTI systems are well understood and they can be used to approximate many real systems.

The impulse response $h[n]$ of the "ideal" interpolator is

$$h[n] = \frac{\sin[\pi(n-\tau)]}{\pi(n-\tau)} = sinc(n-\tau) \tag{4.1}$$

where τ is a time-shift.

From the impulse response $h[n]$ we can plot the frequency response $H(e^{j\omega})$ as shown below (Figure 4.1).

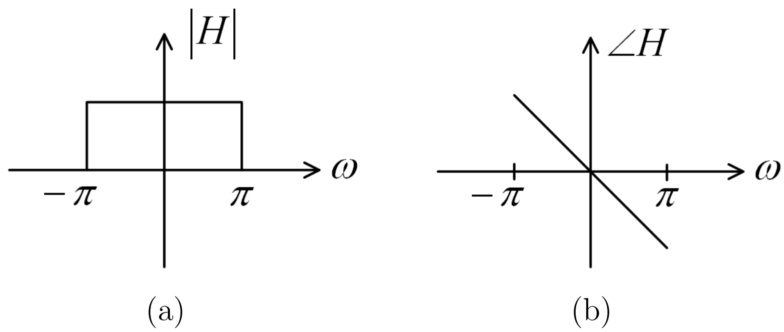

Figure 4.1. Frequency response of an ideal interpolator

4.2.2 "Physical" Sampler

Now, suppose that we have a continuos-to-discrete-time (C/D) converter as shown below. The two principal inputs of the converter are the signal $x(t)$ that we want to sample and the

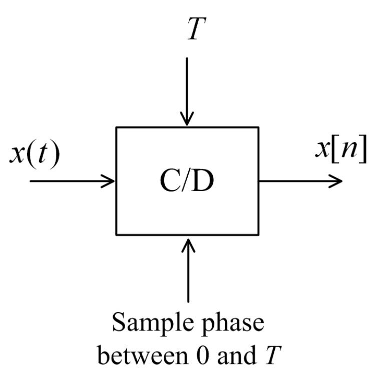

Figure 4.2. Continuos-to-discrete-time converter

sampling period T. Another important input is the sampling phase (always between 0 and T). This number represents the phase between the input and output of the C/D converter. Ideally, the sampling phase must be zero.

To understand the frequency response of a signal that has been sampled and then interpolated, consider the system shown in Figure 4.3. The effect of sampling the continuos signal $x(t)$ can be modeled as a product of $x(t)$ and an impulse train $s(t)$ in the time domain, where

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

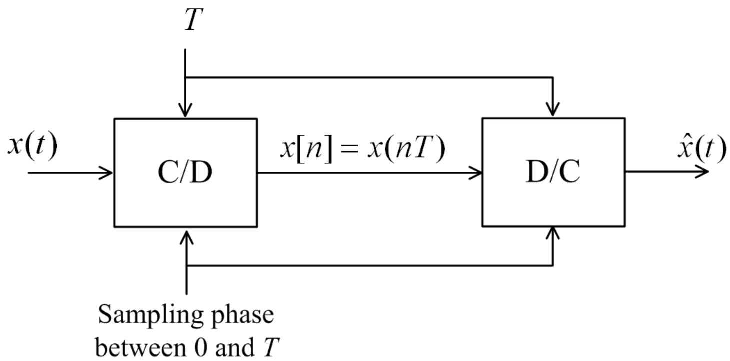

Figure 4.3.

Our interest is to analyze the product $x(t)s(t)$ in the frequency domain.

4.2.3 $x(t)s(t)$ in frequency domain

Recall that multiplication in time domain corresponds to convolution in frequency domain multipled by a scaling factor of $\frac{1}{2\pi}$. Using this fact it is possible to show that the Fourier transform of $s(t)$ is

$$S(j\Omega) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{T}\delta(\Omega - n\Omega_s)$$

where

$$\Omega_s = \frac{2\pi}{T}$$

4.2.4 Since interpolation in the frequency domain

As an example, let $x(t)$ be a signal with the following frequency contents:

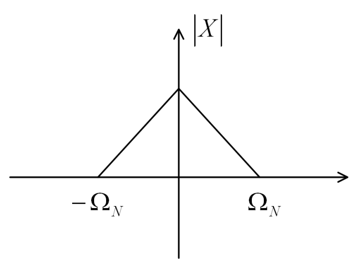

Figure 4.4. Frequency contents of $x(t)$

If we sample $x(t)$ at frequency Ω_s then the frequency response of the sampled signal is

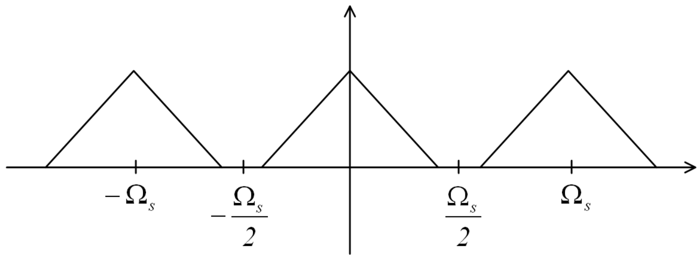

Figure 4.5. Frequency contents of the sampled version of $x(t)$

Note that in order to prevent aliasing and recover the original signal, the sampling frequency Ω_s must be greater than $2\Omega_N$. If this condition is satisfied, we can apply the following lowpass filter to recover the original signal.

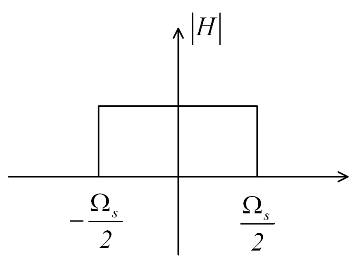

Figure 4.6. Ideal Lowpass filter

4.3 DTFT properties review

Given a real or complex sequence $x[n]$, the discrete-time Fourier transform of $x[n]$ is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The following inverse transform recovers the discrete-time sequence from its DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{-j\omega n})e^{j\omega n}d\omega$$

4.3.1 Linearity

If

$$x_1[n] \Leftrightarrow X_1(e^{j\omega})$$

and

$$x_2[n] \Leftrightarrow X_2(e^{j\omega})$$

then

$$\alpha x_1[n] + \beta x_2[n] \Leftrightarrow \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$

4.3.2 Convolution in time domain

If

$$x[n] \Leftrightarrow X(e^{j\omega})$$

and

$$h[n] \Leftrightarrow H(e^{j\omega})$$

then

$$x[n] * h[n] \Leftrightarrow X(e^{j\omega})H(e^{j\omega})$$

4.3.3 Modulation in time domain

If

$$x[n] \Leftrightarrow X(e^{j\omega})$$

and

$$w[n] \Leftrightarrow W(e^{j\omega})$$

then

$$x[n]w[n] \Leftrightarrow \frac{1}{2\pi} X(e^{j\omega}) * W(e^{j\omega})$$

4.3.4 Real signal

if $x[n]$ is a real signal then its DTFT $X(e^{j\omega})$ is "conjugate symmetric".

4.4 Z-transform review

One problem with the DTFT is that it does not converges absolutely for many simple sequences (e.g. increasing exponentials). The necessity to allow "unstable" signals/systems motivates the need of the Z-transform.

First, we start with the DTFT of a sequence.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

We then introduce a term *alpha* to help improve the convergence of the transform.

$$X(\alpha e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \alpha^{-n} e^{-j\omega n}$$

We use alpha to control otherwise diverging sums. If we let $z = \alpha e^{j\omega}$ we obtain the definition of the Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

The set of $z \in \mathbb{C}$ values for which the Z-transform converges is called the region of convergence (R.O.C.).