EE 123: Digital Signal Processing	**Spring 2007**

Lecture 5 — January 25, evening

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5.1 Outline

These notes cover the following topics:

- z-transform and Region of Convergence
- Poles and Zeros
- z-transform and Region of Convergence Properties
- Inverting z-Transforms Using Partial Fraction Expansion
- Inverting Irrational z-transforms

5.2 z-transform and Region of Convergence

5.2.1 z-tranform definition

Like the DTFT, the z-transform is a tool for representing and analyzing sequences. However, the z-transform is a more general representation because it converges for a broader class of sequences. It is defined:

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

The mapping between a sequence and its z-transform is denoted by:

$$x[n] \overset{Z}{\longleftrightarrow} X(z)$$

This sum is very similar in form to the DTFT. In fact, the z-transform is the DTFT of a sequence where the n-th entry has been multiplied by the real number $|z|^n$ and the frequency ω is associated with the complex part so that $z = |z|e^{j\omega}$. When $|z| = 1$, the z-tranform is equivalent to the DTFT.

5.2.2 Convergence of the z-transform

Convergence for the z-transform means that the infinite sum is absolutely summable, or:

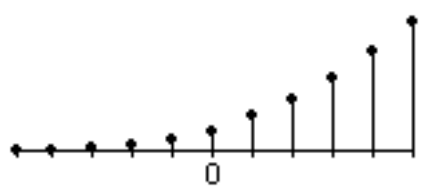
$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

Just as the DTFT does not converge for all sequences, the z-transform does not converge for all sequences and all values of $|z|$. However, the z-transform converges for sequences for which the DTFT does not. As we shall see, the complex sequence z^n can be used to force convergence. The set of values of $|z|$ for which the z-transform converges is called the region of convergence (ROC).

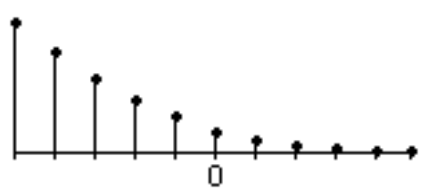
If z_1 is in the ROC, then all z such that $|z| = |z_1|$ are also in the ROC. This is a consequence of the fact that only the absolute value of z determines convergence in the sum. When the ROC is plotted in the complex z-plane, this means that points in the ROC form circles centered about the origin. If one of the circles is the unit circle, then the DTFT exists for that sequence.

Example 1

Let us try to find the z-transform for a^n, where $a > 1$. This is an exponentially growing sequence:



Clearly, the problem with convergence is due to the right side. Intuitively, we need to multiply the sequence by a z^n such that the right side will be pushed down into convergence. If we choose z such that the sum for n positive is forced to converge, we need $|z| > a$. However, the negative end of the sequence gets multiplied by positive powers of z. Now we no longer have convergence for the left side:



The solution is to choose only one side of the sequence to be represented by the z-transform. This corresponds to multiplying by either $u[n]$ or $-u[-n-1]$. $u[n]$ produces the right-sided, or causal, sequence, while $-u[-n-1]$ gives the left-sided, or anti-causal, sequence. Here, the right-sided sequence is chosen:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

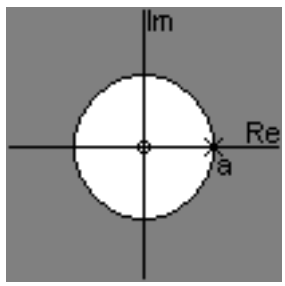
For convergence, we need:

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

This is a geometric series that only converges when $|az^{-1}| < 1$, or $|z| < a$. The formula for a geometric series sum gives:

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > a$$

The ROC is the entire plane outside the circle $|z| = a$:



In general, a causal sequence will have an ROC that extends to ∞.

Example 2

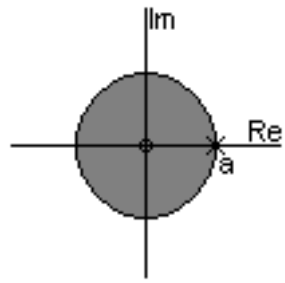
The z-transform of the anti-causal part of the sequence, $-a^n u[-n-1]$, can be found in the same way:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

This geometric series converges when $|z| < a$:

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| < a$$

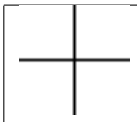
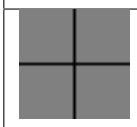
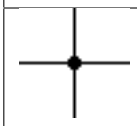
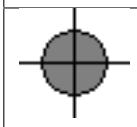
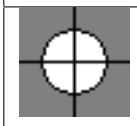
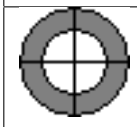
The z-transform expression is the same as for example 1. However, the region of convergence is opposite; the ROC is formed by the inside of a circle:



Generally, an anti-causal sequence will have an ROC that includes 0.

5.2.3 Possibilities for ROC shape

There are six shapes that the ROC will take:

	Null set
	Everywhere
	Point at zero or infinity
	Inside a circle
	Outside a circle
	Inside two circles/Annulus

An annulus may be seen as the intersection of an ROC inside a circle and an ROC lying outside a circle. An annulus may extend to, but not include, zero or infinity.

5.3 Poles and Zeros

On these ROC plots are indicated the poles (X's) and zeros (O's). Poles are z-values are the roots of the denominator, where $\frac{1}{X(z)} = 0$. Zeros are the roots of the numerator, where $X(z) = 0$. The location of poles and zeros does not depend on the ROC. However, ROC's may not include poles. In addition, ROC's are usually bounded by poles.

5.4 z-transform and ROC Properties

A few z-transform properties will be discussed here.

5.4.1 Linearity

For two sequences and their associated z-transforms and ROC's,

$$x_1[n] \overset{Z}{\longleftrightarrow} X_1(z), \quad ROC = R_{x1}$$

$$x_2[n] \overset{Z}{\longleftrightarrow} X_2(z), \quad ROC = R_{x2}$$

the linearity property states:

$$ax_1[n] + bx_2[n] = aX_1(z) + bX_2(z), \quad ROC\, contains\, R_{x1} \cap R_{x2}$$

What happens to the poles when two sequences are added? Usually, the set of poles is $Poles(X_1) \cap Poles(X_2)$.
Zeros, on the other hand, cannot be determined without simplifying the z-transform. An example illustrating this follows:

Example 3

Let's find the z-transform of:

$$x[n] = (\frac{1}{2})^n u[n] + (-\frac{1}{3})^n u[n]$$

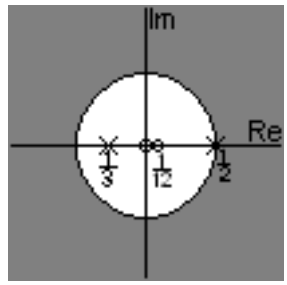
From the first example and linearity, we get:

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

Here it obvious that both exponential sequences have a zero at zero. However, this is not the case of the sum of the sequences, $x[n]$:

$$X(z) = \frac{2 - \frac{1}{6}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

While the poles are the union of the poles of the individual terms, the zeros appear at $z = 0, \frac{1}{12}$. Because the sequence is causal, the ROC extends from the outer pole to infinity.



5.4.2 Cases with ROC everywhere

It is interesting to note the classes of sequences that converge for the entire z-plane:

1. Finite sequences
2. Gaussians: $\binom{m}{r}_q = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}$

5.4.3 Time-shifting property

$x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z), \quad ROC = R_x \, (except\, for\, the\, possible\, addition\, or\, deletion\, of\, z = 0\, or\, z = \infty)$

5.4.4 Differentiation property and higher-order poles

The differentiation property states:

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \quad ROC = R_x$$

We can use this fact to find the z-transform of $x[n] = na^n u[n] = n(a^n u[n])$:

$$X(z) = -z\frac{d}{dz}(\frac{1}{1-az^{-1}}) = \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

The repeated roots in the denominator appear as double poles when plotted.

5.4.5 Convolution of sequences

$$x_1[n] * x_2[n] = X_1(z)X_2(z), \quad ROC\, contains\, R_{x1} \cap R_{x2}$$

5.5 Inverting z-transforms

5.5.1 Long division

To obtain the left-sided sequence, simply divide with the divisor expressed in powers of z. The right-sided sequence is calculated by dividing by a divisor in powers of z^{-1}:

Example 4

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & 1 - az^{-1} \\
\hline
 & \quad az^{-1} \\
 & \quad az^{-1} - a^2z^{-2} \\
\hline
 & \qquad a^2z^{-2} \cdots
\end{array}
$$

The sequence can be read off from the constant coefficients of each term. In this example, $x[0] = 1$, $x[1] = a$, $x[2] = a^2$... Then $x[n]$ must be inferred from a partial sequence. This method is not preferred because long division must be performed until a pattern is recognized.

5.5.2 Partial fraction expansion

By expanding the polynomial into factors of the denominator, we can match the terms to known z-transform pairs. If the order of the numerator (n) is less than the order of the denominator, and the poles are all first order, then this simplifies to a sum of terms with constant numerators:

$$\frac{N(z)}{D(z)} = \sum_{k=1}^{n} \frac{A_k}{1 - d_k z^{-1}}$$

The constant coefficients can be found by multiplying the LHS by the term's denominator and evaluating at $z = d_k$.

$$A_k = (1 - d_k z^{-1}) X(z)|_{z=d_k}$$

5.5.3 Inverting irrational z-transforms

Contour integral

We can directly evaluate the synthesis equation, which unfortunately contains a contour integral:

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz, \quad over\, any\, |z|\, in\, ROC$$

Power Series Expansion

If we can express the function as a power series containing z^{-n}, then we can read off the sequence directly from the inside of the summation:

Example 5

$$X(z) = \ln(1 + az^{-1})$$

Expanding by power series:

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(az^{-1})^n = \sum_{n=1}^{\infty}(\frac{(-1)^{n+1}}{n}a^n)z^{-n}$$

By grouping the terms this way, it is easy to see that this power series is exactly the z-transform of $x[n] = \frac{(-1)^{n+1}}{n}a^n u[n-1]$.

5.5.4 Evaluating for one value of n

If we want to evaluate $n[0]$, and we have $x[n] = 0$ for $n < 0$, we can assume that all infinities in the ROC act as a single point, and evaluate $X(z)$ as $z \to \infty$:

$$x[0] = \lim_{z\to\infty} X(z)$$

If we want to evaluate at $x[1]$, we can shift in frequency domain by z:

1. Subtract $x[0]$
2. Shift back by 1 in time: multiply by z in frequency
3. Take limit as $z \to \infty$

This process can be iteratively repeated for $n > 1$.