Problem Set 3

EECS123: Digital Signal Processing

Prof. Ramchandran
Spring 2008

1. Let $x[n]$ be a finite real sequence, which is non-zero only for $|n| \leq M$. Let

$$
\bar{x} = \frac{1}{2M+1} \sum_{n=-M}^{M} x[n], \quad P_x = \frac{1}{2M+1} \sum_{n=-M}^{M} x^2[n], \text{ and } \sigma^2_x = \frac{1}{2M+1} \sum_{n=-M}^{M} (x[n] - \bar{x})^2.
$$

(a) Show that $\sigma^2_x = P_x - (\bar{x})^2$.

(b) Let,

$$
E(x[n], c) = \frac{1}{2M+1} \sum_{n=-M}^{M} (x[n] - c)^2, \quad c \in \mathbb{R}.
$$

Show that $E(x[n], c)$ is minimized at $c = \bar{x}$.

(c) Write a formula for $\sigma^2_x$ in terms of $X(e^{j\omega})$.

**Note:** $\sigma^2_x$ can be thought of as the mean-square error when $x[n]$ is approximated by a constant sequence $\bar{x}$.

2. Consider the frequency response (DTFT) $H(e^{j\omega})$ of a discrete-time LTI system. Let $h[n]$ be the impulse response. Assume that $h[n]$ satisfies the following five properties:

(i) The system is causal.

(ii) $H(e^{j\omega}) = H^*(e^{-j\omega})$.

(iii) The DTFT of the sequence $h[n + 1]$ is real.

(iv) $\frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega = 2$.

(v) $H(e^{j\pi}) = 0$.

Answer the following questions:

(a) Show that properties (i)–(iii) imply that $h[n]$ is non-zero for only a finite duration.

(b) Find all possible discrete signals $h[n]$ that satisfy properties (i)–(v).
3. Let $0 < |a| < |b|$. Find the inverse $Z$-transform of

$$X(z) = \log\left(\frac{z + a}{z + b}\right), \ |z| > |b|,$$

where log is the natural log or log to the base $e$.

4. Determine the region of convergence of $Y(z)$ where,

(a) $Y(z) = X_1(z) + X_2(z),

\begin{align*}
X_1(z) &= \frac{z}{z^2 + 1}, \quad |z| > 1 \\
X_2(z) &= \frac{z^2}{z + 1}, \quad |z| > 1.
\end{align*}$

(b) $Y(z) = H(z)X(z),

\begin{align*}
x[n] &= \delta[n] + 2\delta[n - 1] \\
H(z) &= \frac{1}{z^2 + 7z + 10}, \quad |z| > 5.
\end{align*}$

(c) $Y(z) = z^{-2}X(z),

\begin{align*}
x[n] &= 2^n u[-n - 1].
\end{align*}$

5. Suppose that the 8-point DFT of a sequence $\{x_n\}_{n=0}^7 = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ is given by $\{X_m\}_{m=0}^7 = \{A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$. If the 4-point DFT of another sequence $\{y_n\}_{n=0}^3 = \{b_0, b_1, b_2, b_3\}$ is given by $\{Y_m\}_{m=0}^3 = \{A_0, A_2, A_4, A_6\}$, find the $b_k$’s in terms of the $a_k$’s.

6. Determine $z_n$, the cyclic convolution of $x_n$ and $y_n$ for the following cases:

(a) $\{x_n\}_{n=0}^5 = \{1, 2, 3, 4, 5, 6\}$ and $\{y_n\}_{n=0}^5 = \{1, 0, 0, 1, 0, 0\}$.

(b) $\{x_n\}_{n=0}^8 = \{1, 2, 3, 4, 5, 6, 0, 0\}$ and $\{y_n\}_{n=0}^8 = \{1, 0, 0, 1, 0, 0, 0, 0\}$. 