Today

- Last time:
  - DTFT - Ch 2
- Today:
  - Continue DTFT
  - Z-Transform briefly!
  - Ch. 3

Properties of the DTFT cont.

**Time-Reversal**

\[ x[n] \leftrightarrow X(e^{i\omega}) \]
\[ x[-n] \leftrightarrow X(e^{-i\omega}) \]

= \( X^*(e^{j\omega}) \) if \( x[n] \in \text{Real} \)

If \( x[n] = x[-n] \) and \( x[n] \) is real, then:

\[
X(e^{j\omega}) = X^*(e^{j\omega})
\]
\[ \rightarrow X(e^{j\omega}) \in \text{Real} \]

Q: Suppose:

\[
x[n] \leftrightarrow X(e^{j\omega})
\]

\[ ? \leftrightarrow Re \{X(e^{j\omega})\} \]

A: Decompose \( x[n] \) to even and odd functions

\[
x[n] = x_e[n] + x_o[n]
\]

\[
x_e[n] := \frac{1}{2}(x[n] + x[-n])
\]
\[
x_o[n] := \frac{1}{2}(x[n] - x[-n])
\]

\[ x_e[n] + x_o[n] \rightarrow Re \{X(e^{j\omega})\} + jIm \{X(e^{j\omega})\} \]
Properties of the DTFT cont.

Time-Freq Shifting/modulation:

\[ x[n] \leftrightarrow X(e^{j\omega}) \]

\[ x[n - nd] \leftrightarrow e^{-j\omega nd} X(e^{j\omega}) \]

\[ e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)}) \]

Example 2

What is the DTFT of:

\[ e^{j\pi n} \]

High Pass Filter

See 2.9 for more properties
Frequency Response of LTI Systems

Check response to a pure frequency:

\[ e^{j\omega n} \rightarrow \text{LTI} \rightarrow y[n] \]

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} \]

\[ = \left( \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n} \]

\[ H(e^{j\omega}) \big|_{\omega=\omega_0} \]

Output is the same pure frequency, scaled and phase-shifted!

\[ e^{j\omega_0 n} \] is an eigen function of LTI systems

Recall eigen vectors satisfy: \( A\nu = \lambda\nu \)

Example 3

Frequency response of a causal moving average filter

\[ y[n] = x[n-M] + \cdots + x[n] \]

\[ = \frac{x[n-M] + \cdots + x[n]}{M+1} \]

Q: What type of filter is it? A: Low-Pass

\[ h[n] = \frac{1}{M+1} w[n - \frac{M}{2}] \]

Same as example 1, only: Shifted by N, divided by M+1, M=2N

\[ H(e^{j\omega}) = \frac{e^{-j\omega \frac{M}{2}}}{M+1} \cdot \frac{\sin \left( \left( \frac{M}{2} + \frac{1}{2} \right) \omega \right)}{\sin \left( \frac{\omega}{2} \right)} \]
Example 3 Cont.
Frequency response of a causal moving average filter

\[ H(e^{j\omega}) = e^{-j\omega \frac{M}{2}} \cdot \frac{\sin\left(\frac{M}{2} + 1\right)\omega}{M + 1} \cdot \frac{\sin\left(\frac{\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \]
Not a sinc!

Example 4:
Impulse Response of an Ideal Low-Pass Filter

\[ h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega \]
\[ = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]
\[ = \frac{1}{2\pi jn} e^{j\omega n} \bigg|_{\omega_c}^{\omega_c} = 2j \sin(w_c n) \]
\[ = \frac{\sin(w_c n)}{\pi n} \]
Non causal! Truncate and shift right to make causal

Example 4
Impulse Response of an Ideal Low-Pass Filter

\[ h_{LP}[n] = \frac{\sin(w_c n)}{\pi n} \]
sampled “sinc”
Example 4

Impulse Response of an Ideal Low-Pass Filter

Non causal! Truncate and shift right to make causal

How does it changes the frequency response?

Truncation:
\[ \tilde{h}_{LP}[n] = w_N[n] \cdot h_{LP}[n] \]

property 2.9.7:
\[ \tilde{H}_{LP}(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \]

Periodic convolution

The z-Transform

• Used for:
  – Analysis of LTI systems
  – Solving difference equations
  – Determining system stability
  – Finding frequency response of stable systems

Eigen Functions of LTI Systems

• Consider an LTI system with impulse response \( h[n] \):

\[ x[n] \xrightarrow{\text{LTI}} y[n] \]

• We already showed that are eigen-functions

\[ x[n] = e^{j\omega n} \]

• What if \( x[n] = z^n = re^{j\omega n} \)
Eigen Functions of LTI Systems

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = \left( \sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n = H(z) z^n \]

- \( x[n] = z^n \) are also eigen-functions of LTI Systems
- \( H(z) \) is called a transfer function
- \( H(z) \) exists for larger class of \( h[n] \) than \( H(e^{j\omega}) \)

The z Transform

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

- Since \( z = re^{j\omega} \)

\[ X(z)|_{z = e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \text{DTFT} \{ x[n] \} \]

Region of Convergence (ROC)

- The ROC is a set of values of \( z \) for which the sum

\[ \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

Converges.

Region of Convergence (ROC)

- Example 1: Right-sided sequence \( x[n] = a^n u[n] \)

\[ X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \]

Recall:

\[ 1 + x + x^2 + \cdots = \frac{1}{1-x}, \text{ if } |x| < 1 \]

So:

\[ X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC} = \{ z : |z| > |a| \} \]
Region of Convergence (ROC)

• Example 2: \( x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \)

\[
X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}}
\]

ROC = \{ z : |z| > \frac{1}{2} \} \cap \{ z : |z| > \frac{1}{3} \}

= \{ z : |z| > \frac{1}{2} \}

• Expression is the same as Example 1!
• ROC = \{ z : |z| < |a| \} is different

The z-transform without ROC does not uniquely define a sequence!

Region of Convergence (ROC)

• Example 3: Left sided sequence \( x[n] = -a^n u[-n - 1] \)

\[
X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{m=1}^{\infty} -a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m
\]

if \( |a^{-1}|z < 1 \), i.e., \( |z| < |a| \) then,

\[
X(z) = \frac{1}{1 - \frac{1}{a-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}
\]

Region of Convergence (ROC)

• Example 4: \( x[n] = -\left(\frac{1}{2}\right)^n u[-n - 1] + \left(-\frac{1}{3}\right)^n u[n] \)

\[
X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}}
\]

ROC = \{ z : |z| < \frac{1}{2} \} \cap \{ z : |z| > \frac{1}{3} \}

= \{ z : \frac{1}{3} < |z| < \frac{1}{2} \}

Same as example 2
Region of Convergence (ROC)

- Example 5: \( x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n - 1] \)
  
  \[
  \text{ROC} = \{ z : |z| > \frac{1}{2} \} \cap \{ z : |z| < \frac{1}{3} \} \\
  = 0
  \]

- Example 6: \( x[n] = a^n \), two sided \( a \neq 0 \)
  
  \[
  \text{ROC} = \{ z : |z| > a \} \cap \{ z : |z| < a \} \\
  = 0
  \]

Region of Convergence (ROC)

- Example 7: Finite sequence \( x[n] = a^n u[n]u[-n + M - 1] \)
  
  \[
  X[z] = \sum_{n=0}^{M-1} a^n z^{-n} \quad \text{Finite, always converges} \\
  = \frac{1 - a^M z^{-M}}{1 - az^{-1}} \quad \text{Zero cancels pole} \\
  = \prod_{k=1}^{M-1} (1 - ae^{j\frac{2\pi k}{M}} z^{-1}) \\
  \text{ROC} = \{ z : |z| > 0 \}
  \]

Properties of ROC

- A ring or a disk in Z-plane, centered at the origin
- DTFT converges iff ROC includes the unit circle
- ROC can’t contain poles
Properties of ROC

- For finite duration sequences, ROC is the entire z-plane, except possibly z=0, z=∞

\[
X(z) = 1 + z^{-1} + z^{-2} \quad \text{ROC excludes } z = 0
\]

\[
X(z) = 1 + z^{1} + z^{2} \quad \text{ROC excludes } z = \infty
\]

Properties of the ROC

- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  Examples 1,2
- For left-sided: inwards from inner most pole to zero
  Example 3
- For two-sided, ROC is a ring - or do not exist
  Examples 4,5,6

Several Properties of the Z-transform

\[
x[n - n_d] \leftrightarrow z^{-n_d} X(z)
\]
\[
z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)
\]
\[
nx[n] \leftrightarrow -z \frac{dX(z)}{dz}
\]
\[
x[-n] \leftrightarrow X(z^{-1})
\]
\[
x[n] * y[n] \leftrightarrow X(z)Y(z)
\]

Inversion of the z-Transform

- In general, by contour integration within the ROC

\[
x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}
\]

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Power series expansion
  - Partial fraction expansion
  - Residue theorem

- Most useful is the inverse of rational polynomials

\[
X(z) = \frac{B(z)}{A(z)} \quad \text{Why?}
\]