Announcements

• Last time:
  – FFT
• Today Frequency Analysis with DFT
• Read Ch. 10.1-10.2
• Who started playing with the SDR?

Spectral Analysis with the DFT

The DFT can be used to analyze the spectrum of a signal.

It would seem that this should be simple, take a block of the signal and compute the spectrum with the DFT.

However, there are many important issues and tradeoffs:
  • Signal duration vs spectral resolution
  • Signal sampling rate vs spectral range
  • Spectral sampling rate
  • Spectral artifacts

What is this?

The first NMR spectrum of ethanol 1951.
Spectral Analysis with the DFT

Consider these steps of processing continuous-time signals:

- Antialiasing lowpass filter
- Continuous-to-discrete-time conversion
- DFT

Parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling interval</td>
<td>(T)</td>
<td>s</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>(\Omega_s)</td>
<td>(\frac{2\pi}{T}) rad/s</td>
</tr>
<tr>
<td>Window length</td>
<td>(L)</td>
<td>unitless</td>
</tr>
<tr>
<td>Window duration</td>
<td>(L \cdot T)</td>
<td>s</td>
</tr>
<tr>
<td>DFT length</td>
<td>(N \cdot T)</td>
<td>(\Omega_c) rad/s</td>
</tr>
<tr>
<td>DFT duration</td>
<td>(\Omega_c)</td>
<td>(\frac{2\pi}{N \cdot T}) rad/s</td>
</tr>
<tr>
<td>Spectral resolution</td>
<td>(\Omega_c)</td>
<td>(\frac{2\pi}{L \cdot T}) rad/s</td>
</tr>
<tr>
<td>Spectral sampling interval</td>
<td>(\Omega_s)</td>
<td>(\frac{2\pi}{N \cdot T}) rad/s</td>
</tr>
</tbody>
</table>

Filtered Continuous-Time Signal

We consider an example:

\[
x_c(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t
\]

\[
X_c(j\Omega) = A_1 \pi [\delta(\Omega - \omega_1) + \delta(\Omega + \omega_1)] + A_2 \pi [\delta(\Omega - \omega_2) + \delta(\Omega + \omega_2)]
\]

Sampled Filtered Continuous-Time Signal

Sampled Signal

If we sampled the signal over an infinite time duration, we would have:

\[
x[n] = x_c(t)|_{t=nT}, \quad -\infty < n < \infty
\]

described by the discrete-time Fourier transform:

\[
X(e^{j\Omega T}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_c \left( f \left( \Omega - r \frac{2\pi}{T} \right) \right), \quad -\infty < \Omega < \infty
\]

Recall \(X(e^{j\omega}) = X(e^{j\Omega T})\), where \(\omega = \Omega T\) ... more in ch 4.
Sampled Filtered Continuous-Time Signal

In the examples shown here, the sampling rate is $\Omega_s/2\pi = 1/T = 20$ Hz, sufficiently high that aliasing does not occur.

![Sampled Signal and DTFT](image)

Windowed Sampled Signal

**Windowed Block of $L$ Signal Samples**

We take the block of signal samples and multiply by a window of duration $L$, obtaining:

$$v[n] = x[n] \cdot w[n], \quad 0 \leq n \leq L - 1$$

Suppose the window $w[n]$ has DTFT $W(e^{j\omega})$.

Then the windowed block of signal samples has a DTFT given by the periodic convolution between $X(e^{j\omega})$ and $W(e^{j\omega})$:

$$V(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

**Block of $L$ Signal Samples**

In any real system, we sample only over a finite block of $L$ samples:

$$x[n] = x_c(t)|_{t = nT}, \quad 0 \leq n \leq L - 1$$

This simply corresponds to a rectangular window of duration $L$.

Recall: in Homework 1 we explored the effect of rectangular and triangular windowing

Windowed Sampled Signal

Convolutions with $W(e^{j\omega})$ have two effects in the spectrum:

- It limits the spectral resolution. – Main lobes of the DTFT of the window
- The window can produce spectral leakage. – Side lobes of the DTFT of the window

* These two are always a tradeoff - time-frequency uncertainty principle
### Windows (as defined in MATLAB)

<table>
<thead>
<tr>
<th>Name(s)</th>
<th>Definition</th>
<th>MATLAB Command</th>
<th>Graph (M = 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>( w[n] = \begin{cases} 1 &amp; \text{if }</td>
<td>n</td>
<td>\leq M/2 \ 0 &amp; \text{if }</td>
</tr>
<tr>
<td>Triangular</td>
<td>( w[n] = \begin{cases} 1 - \frac{</td>
<td>n</td>
<td>}{M/2 + 1} &amp; \text{if }</td>
</tr>
<tr>
<td>Bartlett</td>
<td>( w[n] = \begin{cases} \frac{M}{M/2 + 1} &amp; \text{if }</td>
<td>n</td>
<td>\leq M/2 \ 0 &amp; \text{if }</td>
</tr>
</tbody>
</table>

All of the window functions \( w[n] \) are real and even.

All of the discrete-time Fourier transforms

\[
W(e^{j\omega}) = \sum_{n=-M}^{M} w[n]e^{-jn\omega}
\]

are real, even, and periodic in \( \omega \) with period \( 2\pi \).

In the following plots, we have normalized the windows to unit d.c. gain:

\[
W(e^{j0}) = \sum_{n=-M}^{M} w[n] = 1
\]

This makes it easier to compare windows.

### Window Example

- For \( M = 16 \):
  - **Bartlett**
  - **Triangular**
  - **Hamming**

  ![Graphs](graphs_16)

- For \( M = 256 \):
  - **Bartlett**
  - **Triangular**
  - **Hamming**

  ![Graphs](graphs_256)