Last Time

- Started with STFT
- Heisenberg Boxes
- Continue and move to wavelets
- Ham -- Get me the forms!

DFT

\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \]

- \( \Delta \omega = \frac{2\pi}{N} \)
- \( \Delta t = N \)
- \( \Delta \omega \cdot \Delta t = 2\pi \)

one DFT coefficient

Discrete STFT

\[ X[r, k] = \sum_{m=0}^{L-1} x[rR + m]w[m] e^{-j2\pi km/N} \]

- \( \Delta \omega = \frac{2\pi}{L} \)
- \( \Delta t = L \)

one STFT coefficient
Limitations of Discrete STFT

- Need overlapping $\Rightarrow$ Not orthogonal
- Computationally intensive $O(MN \log N)$
- Same size Heisenberg boxes

From STFT to Wavelets

- Basic Idea:
  - low-freq changes slowly - fast tracking unimportant
  - Fast tracking of high-freq is important in many apps.
  - Must adapt Heisenberg box to frequency

- Back to continuous time for a bit.....

From STFT to Wavelets

- Continuous time

\[
S_f(u, \Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t}dt
\]

\[
W_f(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*(\frac{t-u}{s})dt
\]

*Morlet - Grossmann

The function $\Psi$ is called a mother wavelet
- Must satisfy:
  \[
  \int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1 \quad \Rightarrow \text{unit norm}
  \]
  \[
  \int_{-\infty}^{\infty} \Psi(t) dt = 0 \quad \Rightarrow \text{Band-Pass}
  \]
STFT and Wavelets “Atoms”

STFT Atoms
(with hamming window)

\[ w(t - u)e^{j\Omega u} \]

\[ \Omega_{hi} \]

\[ \Omega_{lo} \]

Wavelet Atoms

\[ \frac{1}{\sqrt{s}} \Psi\left(\frac{t - u}{s}\right) \]

\[ s = 1 \]

\[ s = 3 \]

Examples of Wavelets

- **Mexican Hat**

\[ \Psi(t) = (1 - t^2)e^{-t^2/2} \]

- **Haar**

\[ \Psi(t) = \begin{cases} 
-1 & 0 \leq t < \frac{1}{2} \\
1 & \frac{1}{2} \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]

Example: Wavelet of Chirp

Wavelets VS STFT
Example 2: “Bumpy” Signal

\begin{align*}
\log(s) & \quad \text{SombreroWavelet} \\
u & \quad \text{SombreroWavelet}
\end{align*}

Wavelets Transform

- Can be written as linear filtering

\[
Wf(u, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} f(t) \Psi^* \left( \frac{t - u}{s} \right) dt
\]

\[
= \{ f(t) \ast \overline{\Psi}_s(t) \}(u)
\]

\[
\overline{\Psi}_s = \frac{1}{\sqrt{s}} \Psi \left( \frac{t}{s} \right)
\]

- Wavelet coefficients are a result of bandpass filtering

Wavelet Transform

- Many different constructions for different signals
  - Haar good for piece-wise constant signals
  - Battle-Lemarie': Spline polynomials

- Can construct Orthogonal wavelets
  - For example: dyadic Haar is orthonormal

\[
\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi \left( \frac{t - 2^i n}{2^i} \right)
\]

\[i = [1, 2, 3, \ldots]\]

Orthonormal Haar

- Same scale non-overlapping
- Orthogonal between scales
Scaling function

\[ \overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right) \]

- Problem:
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions

**Recall, for chirp:**

- Solution:
  - Plug low-pass spectrum with a scaling function \( \overline{\Phi} \)

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Haar Scaling function

\[ \Psi(t) = \begin{cases} 
-1 & 0 \leq t < \frac{1}{2} \\
1 & \frac{1}{2} \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]

\[ \Phi(t) = \begin{cases} 
1 & 0 \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]

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Back to Discrete

- Early 80’s, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80’s link to DSP by Daubechies and Mallat.

- From CWT to DWT not so trivial!
- Must take care to maintain properties
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \phi_{s,u}[n] \]

Example: Discrete Haar Wavelet

Haar for n=2

Equivalent to DFT_2!

Discrete Orthogonal Haar Wavelet

Haar for n=8
Fast DWT with Filter Banks

$h_0[n]$  $h_1[n]$  $h_0[n]$  $h_1[n]$

$x[n]$  $h_0[n]$  $a_{0n}$?  $d_{0n}$?

not quite... too many coefficients

complexity:
$N + N/2 + N/4 + N/8 + ... + \frac{N}{2^k} = 2N$
$= O(N)$
Reconstruction

Just flip arrows, replace h with g

Approximation from 25/256 coefficients

Example: Denoising Noisy Signals
Example: Denoising by Thresholding

- Noisy
- Denoised, largest 25 coefficients

Compression - JPEG2000 vs JPEG

- Jpeg2000 - Wavelet
- Jpeg - DCT

@ 66 fold compression ratio
Approximation/Compression

0.000% coefficients
Example in Research

Robust 4D Flow Denoising using Divergence-free Wavelet Transform

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Running head: 4D Flow Denoising with Divergence-free Wavelet Transform

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Noisy Flow Data

Divergence Free Wavelets

- Linear spline $\Phi_0$
- Quadratic spline $\Phi_1$
- Linear spline $\psi_0$
- Quadratic spline $\psi_1$

- Linear Comb. $a$
- Softthresh $\lambda_n$
- Softthresh $\lambda_{df}$
- Linear Comb. $d$