Topics

- Last time
  - Upsampling
  - Resampling by rational fraction
- Today
  - Interchanging Compressors/Expanders with filtering
  - Polyphase decomposition
  - Multi-rate processing

Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π/101 ($$$$$)
  - Downsample by M=101

- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Interchanging Operations

\[ x[n] \xrightarrow{H(z)} \xrightarrow{\uparrow L} y[n] \equiv x[n] \xrightarrow{\uparrow L} H(z^L) \xrightarrow{y[n]} \]

\[ x[n] \xrightarrow{\downarrow M} H(z) \xrightarrow{y[n]} \equiv x[n] \xrightarrow{H(z^M)} \xrightarrow{\downarrow M} y[n] \]
Polyphase Decomposition

- We can decomposed an impulse response to:

\[ h[n] = \sum_{k=0}^{M-1} h_k[n - k] \]

- Define:

\[ h_k[n] \rightarrow \downarrow M \rightarrow e_k[n] \]

\[ e_k[n] = h_k[nM] \]

\[ e_0[n] \rightarrow e_1[n] \rightarrow \cdots \]

- Recall upsampling \( \Rightarrow \) scaling

\[ H_k(z) = E_k(z^M) \]

Also, recall:

\[ h[n] = \sum_{k=0}^{M-1} h_k[n - k] \]

So,

\[ H(z) = \sum_{k=0}^{M-1} E_k(z^M)z^{-k} \]

Why should you care?
**Polyphase Implementation of Decimation**

\[ x[n] \rightarrow H(z) \rightarrow y[n] \rightarrow \downarrow M \rightarrow w[n] = y[nM] \]

- **Problem:**
  - Compute all y[n] and then throw away -- wasted computation!
  - For FIR length N ⇒ N mults/unit time
  - Can interchange Filter with compressor?
    - Not in general!

**Computation:**

Each Filter: \( N/M \times (1/M) \) mult/unit time
Total: \( N/M \) mult/unit time

**What about interpolation?**
Multirate FilterBank

- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
- Often $h_1[n] = e^{j\pi n} h_0[n]$ or $H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$

Subtleties in Time-Freq Tiling

- Assume $h_0$, $h_1$ are ideal low, high pass filters

\[ X(e^{j\omega}) \]
Subtleties in Time-Freq Tiling

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Perfect Reconstruction Ideal Filters

$Y(e^{j\omega}) = \frac{1}{2} \left[ G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega}) \right] X(e^{j\omega})$

$+ \frac{1}{2} \left[ G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)}) \right] X(e^{j(\omega-\pi)})$

need to cancel!

Perfect Reconstruction non-Ideal Filters

Quadrature Mirror Filters - perfect recon

$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$

$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$

$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$
Quadrature Mirror Filters - perfect recon

\[ H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)}) \]
\[ G_0(e^{j\omega}) = 2H_0(e^{j\omega}) \]
\[ G_1(e^{j\omega}) = -2H_1(e^{j\omega}) \]

Example Haar:

\[ h_0[n] \] 
\[ h_1[n] \] 
\[ g_0[n] \] 
\[ g_1[n] \]

Polyphase Filter-Bank

\[ e_{00} = h_0[2n] \]
\[ e_{01} = h_0[2n+1] \]
\[ e_{10} = e_{00}[n] \]
\[ e_{11} = -e_{01}[n] \]