Design Through Optimization

- Idea: Sample/discretize the frequency response

\[ H(e^{j\omega}) \Rightarrow H(e^{j\omega_k}) \]

- Sample points are fixed \( \omega_k = \frac{k\pi}{P} \)
- \( -\pi \leq \omega_1 < \cdots < \omega_p \leq \pi \)
- \( M+1 \) is the filter order
- \( P \gg M + 1 \) (rule of thumb \( P=15M \))
- Yields a (good) approximation of the original problem

Optimality

- Least Squares:

\[
\text{minimize } \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega
\]

Variation: weighted least-squares

\[
\text{minimize } \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega
\]

Example: Least Squares

- Target: Design \( M+1=2N+1 \) filter
- First design non-causal \( \tilde{H}(e^{j\omega}) \) and hence \( \tilde{h}[n] \)
- Then, shift to make causal

\[
h[n] = \tilde{h}[n - M/2]
\]

\[
H(e^{j\omega}) = e^{-j\frac{M}{2}} \tilde{H}(e^{j\omega})
\]
Example: Least Squares

- Matrix formulation:
  \[ \tilde{h} = [\tilde{h}[-N], \tilde{h}[-N+1], \ldots, \tilde{h}[N]]^T \]
  \[ b = [H_d(e^{j\omega_1}), \ldots, H_d(e^{j\omega_P})]^T \]
  \[ A = \begin{bmatrix}
e^{-j\omega_1(-N)} & \ldots & e^{-j\omega_1(N)} \\
                               \vdots \\
e^{-j\omega_P(-N)} & \ldots & e^{-j\omega_P(N)}
\end{bmatrix} \]

\[ \arg\min_{\tilde{h}} ||A\tilde{h} - b||^2 \]

Least Squares

\[ \arg\min_{\tilde{h}} ||A\tilde{h} - b||^2 \]

Solution:
\[ \tilde{h} = (A^*A)^{-1}A^*b \]

- Result will generally be non-symmetric and complex valued.
- However, if \( \tilde{H}(e^{j\omega}) \) is real, \( \tilde{h}[n] \) should have symmetry!

Design of Linear-Phase L.P Filter

- Suppose:
  - \( \tilde{H}(e^{j\omega}) \) is real-symmetric
  - \( M \) is even (\( M+1 \) taps)

- Then:
  - \( \tilde{h}[n] \) is real-symmetric around midpoint

- So:
\[ \tilde{H}(e^{j\omega}) = \tilde{h}[0] + \tilde{h}[1]e^{-j\omega} + \tilde{h}[\omega]e^{j\omega} + \tilde{h}[2]e^{-j2\omega} + \tilde{h}[\omega]e^{j2\omega} + \ldots \]
\[ = \tilde{h}[0] + 2\cos(\omega)\tilde{h}[1] + 2\cos(2\omega)\tilde{h}[2] + \ldots \]

Least-Squares Linear-Phase Filter

Given \( M, \omega_P, \omega_S \) find the best LS filter:

\[ A = \begin{bmatrix}1 & \cdots & 2\cos(M/2\omega_1) \\
                               \vdots \\
1 & \cdots & 2\cos(M/2\omega_P) \\
\end{bmatrix} \]

\[ b = [1, 1, \ldots, 1, 0, 0, \ldots, 0]^T \]
Least-Squares Linear-Phase Filter

Given $M$, $\omega_p$, $\omega_s$ find the best LS filter:

$$A = \begin{bmatrix}
1 & \cdots & 2 \cos \left( \frac{M}{2} \omega_1 \right) \\
\vdots & & \vdots \\
1 & \cdots & 2 \cos \left( \frac{M}{2} \omega_p \right) \\
\vdots & & \vdots \\
1 & \cdots & 2 \cos \left( \frac{M}{2} \omega_s \right)
\end{bmatrix}$$

$$b = [1, 1, \cdots, 1, 0, 0, \cdots, 0]^T$$

$$\tilde{b}_+ = [\tilde{h}[0], \cdots, \tilde{h}[M/2]]^T = (A^* A)^{-1} A^* b$$

$$\tilde{h} = \begin{cases}
\tilde{h}_+[n] & n \geq 0 \\
\tilde{h}_+[-n] & n < 0
\end{cases}$$

$$h[n] = \tilde{h}[n - M/2]$$

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Extension:

- LS has no preference for pass band or stop band
- Use weighting of LS to change ratio

want to solve the discrete version of:

$$\min_\omega \int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

where $W(\omega)$ is $\delta_p$ in the pass band and $\delta_s$ in stop band

Similarly: $W(\omega)$ is 1 in the pass band and $\delta_p/\delta_s$ in stop band

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Weighted Least-Squares

$$\argmin_{\tilde{h}_+} (A\tilde{h}_+ - b)^* W^2 (A\tilde{h}_+ - b)$$

Solution:

$$\tilde{h}_+ = (A^* W^2 A)^{-1} W^2 A^* b$$

$$W = \begin{bmatrix}
1 & & & & \delta_p \\
& 1 & & & \delta_p \\
& & \ddots & & \delta_p \\
& & & \ddots & \delta_p \\
& & & & \delta_p
\end{bmatrix}$$

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Min-Max optimal Filters

- Chebychev Design (min-max)

$$\min_{\omega \in \text{care}} \max |H(e^{j\omega}) - H_d(e^{j\omega})|$$

- Parks-McClellan algorithm - equi-ripple
- Also known as Remez exchange algorithms (signal.remez)
- Also with convex optimization
Specifications

- Filter specifications are given in terms of boundaries

Min-Max Filter Design

- Minimize:
  - max pass-band ripple
    \[ 1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p, \quad 0 \leq \omega \leq \omega_p \]
  - min-max stop-band ripple
    \[ |H(e^{j\omega})| \leq \delta_s, \quad \omega_s \leq \omega \leq \pi \]

Min-max Ripple Design

- Recall, \( \tilde{H}(e^{j\omega}) \) is symmetric and real
- Given \( \omega_p \omega_s M \), find \( \delta, \tilde{h}_+ \):
  - minimize \( \delta \)
  - Subject to:
    \[ 1 - \delta \leq \tilde{H}(e^{j\omega_k}) \leq 1 + \delta, \quad 0 \leq \omega_k \leq \omega_p \]
    \[ -\delta \leq \tilde{H}(e^{j\omega_k}) \leq \delta, \quad \omega_s \leq \omega_k \leq \pi \]
    \[ \delta > 0 \]
- Solution is a linear program in \( \delta, \tilde{h}_+ \)
- A well studied class of problems

Min-Max Ripple via Linear Programming

minimize \( \delta \)
subject to:

\[ 1 - \delta \leq A_p \tilde{h}_+ \leq 1 + \delta \]
\[ -\delta \leq A_s \tilde{h}_+ \leq \delta \]
\[ \delta > 0 \]

\[ A_p = \begin{bmatrix} 1 & 2 \cos(\omega_1) & \cdots & 2 \cos(\frac{M-1}{2} \omega_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 \cos(\omega_p) & \cdots & 2 \cos(\frac{M-1}{2} \omega_p) \end{bmatrix} \]
\[ A_s = \begin{bmatrix} 1 & 2 \cos(\omega_1) & \cdots & 2 \cos(\frac{M-1}{2} \omega_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 \cos(\omega_p) & \cdots & 2 \cos(\frac{M-1}{2} \omega_p) \end{bmatrix} \]
Convex Optimization

- Many tools and Solvers
  - Tools:
    - CVX (Matlab) http://cvxr.com/cvx/
    - CVXOPT, CVXMOD (Python)
  - Engines:
    - Sedumi (Free)
    - MOSEK (commercial)
- Take EE127!

Using CVX (in Matlab)

```matlab
M = 16;
wp = 0.5*pi;
ws = 0.6*pi;
MM = 15*M;
w = linspace(0,pi,MM);
idxp = find(w <=wp);
idxs = find(w >=ws);
Ap = [ones(length(idxp),1) 2*cos(kron(w(idxp),[1:M/2]));
As = [ones(length(idxs),1) 2*cos(kron(w(idxs),[1:M/2]));

% optimization
cvx_begin
variable hh(M/2+1,1);
variable d(1,1);
minimize(d)
subject to
   Ap*hh <=1+d;
   Ap*hh >=1-d;
   As*hh <  d;
   As*hh > -d;
d>0;
cvx_end

h = [hh(end:-1:1) ; hh(2:end)];
```

Variations:

- Convex Problems:
  - Fix $\delta_s$ optimize for $\delta_p$
  - Fix $\delta_p$ optimize for $\delta_s$
  - Linear constraints on $h[n]$
- Quasi-Convex (feasible through bisection)
  - Fix $\delta_p$, $\delta_s$, $M$, minimize $\Delta \omega = \omega_s - \omega_p$
  - Fix $\delta_p$, $\delta_s$, $\Delta \omega = \omega_s - \omega_p$, minimize $M$

Bisection Example: Minimize $M$

- given $\delta_p$, $\delta_s$, $\Delta \omega = \omega_s - \omega_p$ Initialize problem with:
  - Set $M_{\text{min}}$ to be small and hence infeasible
  - Set $M_{\text{max}}$ to be large and hence feasible
  - Set $M = \text{floor}(M_{\text{max}}/2 + M_{\text{min}}/2)$
- Given $M$, $\delta_p$, $\Delta \omega = \omega_s - \omega_p$ solve for minimum $\delta_s$
  - If $\delta_s$ violates constrains, set $M_{\text{min}} = M$
  - if $\delta_s$ within constraints, set $M_{\text{max}} = M$
  - Set $M = \text{floor}(M_{\text{max}}/2 + M_{\text{min}}/2)$
  - Repeat till $M$ is tight
IIR Design

• Historically
  – Continuous IIR design was advanced
  – Use results from C.T to D.T
  – C.T IIR designs have closed form, easy to use
  – Easy to control Magnitude, not easy to control phase

• Common Types:
  – Butterworth - monotonic, no ripple
  – Chebyshev - Type I, pass band ripple, Type II stop band ripple
  – Elliptic - Ripples in both bands

Design of D.T IIR Filters from Analog

• Discretize by one of many techniques
• $H_c(s) \Rightarrow H(z)$

• Must satisfy:
  – Imaginary axis is mapped to unit circle
  – Stability of $H_c(s)$ should result in stable $H(z)$

• Two methods:
  – Impulse invariance - match impulse response
  – Bilinear transformation