Lecture 6
Properties of DFT

Announcements
• HW1 solutions posted -- self grading due
• HW2 due Friday
• SDR give after GSI Wednesday
• Finish reading Ch. 8, start Ch. 9
• ham radio licensing lectures Tue 6:30-8pm Cory 521

Cool things DSP
• Cosmic Microwave Background radiation

Last Time
• Discrete Fourier Transform
  – Similar to DFS
  – Sampling of the DTFT (subtitles....more later)
  – Properties of the DFT
• Today
  – Linear convolution with DFT
  – Fast Fourier Transform

Properties of DFT
• Inherited from DFS (EE120/20) so no need to be proved
• Linearity
  \[ \alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k] \]
• Circular Time Shift
  \[ x[(n - m)_N] \leftrightarrow X[k] e^{-j(2\pi/N)km} = X[k] W_N^{km} \]

Circular shift
Properties of DFT

• Circular frequency shift
  \[ x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))N] \]

• Complex Conjugation
  \[ x^*[n] \leftrightarrow X^*[((-k))N] \]

• Conjugate Symmetry for Real Signals
  \[ x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))N] \]

Show...

Parseval’s Identity

\[ \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2 \]

Proof (in matrix notation)

\[ x^*x = \left( \frac{1}{N} W_N^* X \right)^* \left( \frac{1}{N} W_N^* X \right) = \frac{1}{N^2} X^* W_N W_N^* X = \frac{1}{N} X^*X \]

Circular Convolution Sum

• Circular Convolution:
  \[ x_1[n] \overset{\bigcirc}{\oplus} x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n - m))N] \]

for two signals of length N

• Note: Circular convolution is commutative
  \[ x_2[n] \overset{\bigcirc}{\oplus} x_1[n] = x_1[n] \overset{\bigcirc}{\oplus} x_2[n] \]

Compute Circular Convolution Sum

Circular ‘flip’ multiply and add
Here: \( y[0] \)

Equivalent periodic convolution over a period

\[ y[n] = x_1[n] \overset{\bigcirc}{\oplus} x_2[n] =? \]
Properties of DFT

- **Circular Convolution:** Let $x_1[n]$, $x_2[n]$ be length $N$

  $$x_1[n] \ast x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

  Very useful! (for linear convolutions with DFT)

- **Multiplication:** Let $x_1[n]$, $x_2[n]$ be length $N$

  $$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \ast X_2[k]$$

Linear Convolution

- **Next....**
  - Using DFT, circular convolution is easy
  - But, **linear** convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Used DFT to do linear convolution

Linear Convolution via Circular Convolution

- **Zero-pad $x[n]$ by P-1 zeros**

  $$x_{zp}[n] = \begin{cases} x[n] & 0 \leq n \leq L - 1 \\ 0 & L \leq n \leq L + P - 2 \end{cases}$$

- **Zero-pad $h[n]$ by L-1 zeros**

  $$h_{zp}[n] = \begin{cases} h[n] & 0 \leq n \leq P - 1 \\ 0 & P \leq n \leq L + P - 2 \end{cases}$$

- **Now, both sequences are of length $M=L+P-1$**
Example

\[ M = L + P - 1 = 8 \]

Example

Linear Convolution using DFT

- In practice we can implement a circulant convolution using the DFT property:

\[
x[n] * h[n] = x_{zp}[n] \odot h_{zp}[n] = \text{DFT}^{-1} \{ \text{DFT} \{ x_{zp}[n] \} \times \text{DFT} \{ h_{zp}[n] \} \}
\]

for \( 0 \leq n \leq M-1, M = L + P - 1 \)

- **Advantage**: DFT can be computed with \( N \log_2 N \) complexity (FFT algorithm later!)
- **Drawback**: Must wait for all the samples -- huge delay -- incompatible with real-time

Block Convolution

- **Problem**:
  - An input signal \( x[n] \), has very long length (could be considered infinite)
  - An impulse response \( h[n] \) has length \( P \)
  - We want to take advantage of DFT/FFT and compute convolutions in blocks that are shorter than the signal

- **Approach**:
  - Break the signal into small blocks
  - Compute convolutions
  - Combine the results
Overlap-Add Method

We decompose the input signal $x[n]$ into non-overlapping segments $x_i[n]$ of length $L$:

$$x_i[n] = \begin{cases} x[n] & nL \leq n \leq (r+1)L-1 \\ 0 & \text{otherwise} \end{cases}$$

The input signal is the sum of these input segments:

$$x[n] = \sum_{r=0}^{\infty} x_i[n]$$

The output signal is the sum of the output segments $x_i[n] * h[n]$:

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} x_i[n] * h[n] \quad (1)$$

Each of the output segments $x_i[n] * h[n]$ is of length $N = L + P - 1$.

Recall:

$\text{Valid linear convolution!}$

Overlap-Save Method

Basic Idea

We split the input signal $x[n]$ into overlapping segments $x_i[n]$ of length $L + P - 1$.

Perform a circular convolution of each input segment $x_i[n]$ with the impulse response $h[n]$, which is of length $P$ using the DFT. Identify the $L$-sample portion of each circular convolution that corresponds to a linear convolution, and save it. This is illustrated below where we have a block of $L$ samples circularly convolved with a $P$ sample filter.

Example of overlap and save: