Spectral Analysis with the DFT

The DFT can be used to analyze the spectrum of a signal.

It would seem that this should be simple, take a block of the signal and compute the spectrum with the DFT.

However, there are many important issues and tradeoffs:
- Signal duration vs spectral resolution
- Signal sampling rate vs spectral range
- Spectral sampling rate
- Spectral artifacts

### Key Parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling interval</td>
<td>( T )</td>
<td>s</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>( \Omega_s = \frac{2\pi}{T} )</td>
<td>rad/s</td>
</tr>
<tr>
<td>Window length</td>
<td>( L )</td>
<td>unitless</td>
</tr>
<tr>
<td>Window duration</td>
<td>( L \cdot T )</td>
<td>s</td>
</tr>
<tr>
<td>DFT length</td>
<td>( N \geq L )</td>
<td>unitless</td>
</tr>
<tr>
<td>DFT duration</td>
<td>( N \cdot T )</td>
<td>s</td>
</tr>
<tr>
<td>Spectral resolution</td>
<td>( \Omega_r = \frac{2\pi}{N T} )</td>
<td>rad/s</td>
</tr>
<tr>
<td>Spectral sampling interval</td>
<td>( \frac{\Omega_r}{\Omega_s} )</td>
<td>rad/s</td>
</tr>
</tbody>
</table>

Filtered Continuous-Time Signal

We consider an example:

\[
x_c(t) = A_1 \cos(\omega_1 t + \omega_2 t) + A_2 \pi \left( \delta(\Omega - \omega_1) + \delta(\Omega + \omega_1) \right) + A_3 \pi \left( \delta(\Omega - \omega_2) + \delta(\Omega + \omega_2) \right).
\]
**Sampled Filtered Continuous-Time Signal**

**Sampled Signal**

If we sampled the signal over an infinite time duration, we would have:

\[ x[n] = x_c(t)_{|t=nT}, \quad -\infty < n < \infty \]

described by the discrete-time Fourier transform:

\[ X(e^{j\Omega}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c \left( j \left( \Omega - \frac{2\pi n}{T} \right) \right), \quad -\infty < \Omega < \infty \]

Recall \( X(e^{j\omega}) = X(e^{j\Omega T}) \), where \( \omega = \Omega T \ldots \) more in ch 4.

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**Windowed Sampled Signal**

**Block of L Signal Samples**

In any real system, we sample only over a finite block of \( L \) samples:

\[ x[n] = x_c(t)_{|t=nT}, \quad 0 \leq n \leq L - 1 \]

This simply corresponds to a rectangular window of duration \( L \).

Recall: in Homework 1 we explored the effect of rectangular and triangular windowing.

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**Windowed Sampled Signal**

Convolution with \( W(e^{j\omega}) \) has two effects in the spectrum:
1. It limits the spectral resolution. – Main lobes of the DTFT of the window
2. The window can produce spectral leakage. – Side lobes of the DTFT of the window

* These two are always a tradeoff - time-frequency uncertainty principle

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**Windows (as defined in MATLAB)**

<table>
<thead>
<tr>
<th>Name(s)</th>
<th>Definition</th>
<th>MATLAB Command</th>
<th>Graph (( f = 0 - 1 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>[ x[n] = \begin{cases} 1 &amp;</td>
<td>n</td>
<td>\leq \frac{M}{2} \ 0 &amp;</td>
</tr>
<tr>
<td>Bartlett</td>
<td>[ x[n] = \begin{cases} 1 &amp;</td>
<td>n</td>
<td>\leq \frac{M}{2} \ \frac{1}{N} &amp;</td>
</tr>
<tr>
<td>Triangular</td>
<td>[ x[n] = \begin{cases} 1 &amp;</td>
<td>n</td>
<td>\leq \frac{M}{2} \ \frac{1}{N} &amp;</td>
</tr>
</tbody>
</table>

---

In the examples shown here, the sampling rate is \( \Omega_s/2\pi = 1/T = 20 \text{ Hz} \). sufficiently high that aliasing does not occur.

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Miki Lustig UCB. Based on Course Notes by J.M Kahn

Spring 2014, EE123 Digital Signal Processing
Windows (as defined in MATLAB)

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
<th>MATLAB Command</th>
<th>Graph (M = 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rect</td>
<td>( c[n] = \frac{1}{2} [1 + \cos(\pi n/M)] )</td>
<td>hanning(M+1)</td>
<td></td>
</tr>
<tr>
<td>Hann</td>
<td>( c[n] = \frac{1}{2} [1 + \cos(\pi n/M)] )</td>
<td>hann(M+1)</td>
<td></td>
</tr>
<tr>
<td>Hamming</td>
<td>( c[n] = [0.5 + 0.5 \cos(\pi n/M)] )</td>
<td>hann(M+1)</td>
<td></td>
</tr>
</tbody>
</table>

Windows

- All of the window functions \( w[n] \) are real and even.
- All of the discrete-time Fourier transforms
  \[
  W(e^{j\omega}) = \sum_{n=-M}^{M} w[n] e^{-jn\omega}
  \]
  are real, even, and periodic in \( \omega \) with period \( 2\pi \).
- In the following plots, we have normalized the windows to unit d.c. gain:
  \[
  W(e^{0}) = \sum_{n=-M}^{M} w[n] = 1
  \]
  This makes it easier to compare windows.

Windows Properties

These are characteristic of the window type

<table>
<thead>
<tr>
<th>Window</th>
<th>Main-lobe ( \delta_{M} )</th>
<th>Sidelobe ( \delta_{S} )</th>
<th>Sidelobe (-20\log_{10} \delta_{S} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rect</td>
<td>( 4\pi / 8\pi )</td>
<td>0.09</td>
<td>21</td>
</tr>
<tr>
<td>Bartlett</td>
<td>( M + 1 / 8\pi )</td>
<td>0.05</td>
<td>26</td>
</tr>
<tr>
<td>Hann</td>
<td>( M + 1 / 8\pi )</td>
<td>0.0063</td>
<td>44</td>
</tr>
<tr>
<td>Hamming</td>
<td>( M + 1 / 12\pi )</td>
<td>0.0022</td>
<td>53</td>
</tr>
<tr>
<td>Blackman</td>
<td>( M + 1 / 16\pi )</td>
<td>0.0002</td>
<td>74</td>
</tr>
</tbody>
</table>

Most of these (Bartlett, Hann, Hamming) have a transition width that is twice that of the rect window.

Warning: Always check what’s the definition of \( M \)

Adapted from A Course In Digital Signal Processing by Buss Parris, Wiley, 1997

Windows Examples

Here we consider several examples. As before, the sampling rate is \( \Omega_s/2\pi = 1/T = 20 \) Hz.

Rectangular Window, \( L = 32 \)

Triangular Window, \( L = 32 \)
Windows Examples

Hamming Window, \( L = 32 \)

Optimal Window: Kaiser

- Minimum main-lobe width for a given side-lobe energy %
  \[
  \frac{\int_{-\infty}^{\infty} |H(e^{j\omega})|^2 d\omega}{\int_{-\infty}^{\infty} |H(e^{j\omega})|^2 d\omega}
  \]
- Window is parametrized with \( L \) and \( \beta \) \( \text{OS Eq 10.12} \)
  - \( \beta \) determines side-lobe level
  - \( L \) determines main-lobe width

Example

Zero-Padding

- In preparation for taking an \( N \)-point DFT, we may zero-pad the windowed block of signal samples to a block length \( N \geq L \):
  \[
  \begin{align*}
  v[n] & \quad 0 \leq n \leq L - 1 \\
  0 & \quad L \leq n \leq N - 1
  \end{align*}
  \]
  - This zero-padding has no effect on the DTFT of \( v[n] \), since the DTFT is computed by summing over \(-\infty < n < \infty\).

Effect of Zero Padding

- We take the \( N \)-point DFT of the zero-padded \( v[n] \), to obtain the block of \( N \) spectral samples:
  \[
  V[k], \quad 0 \leq k \leq N - 1
  \]
Zero-Padding

- Consider the DTFT of the zero-padded $v[n]$. Since the zero-padded $v[n]$ is of length $N$, its DTFT can be written:

$$V(e^{j\omega}) = \sum_{n=0}^{N-1} v[n] e^{-j\omega n}, \quad -\infty < \omega < \infty$$

The $N$-point DFT of $v[n]$ is given by:

$$V[k] = \sum_{n=0}^{N-1} v[n] W_N^{nk}, \quad 0 \leq k \leq N - 1$$

We see that $V[k]$ corresponds to the samples of $V(e^{j\omega})$:

$$V[k] = V(e^{j\omega})|_{\omega = \frac{k}{N} \pi}, \quad 0 \leq k \leq N - 1$$

To obtain samples at more closely spaced frequencies, we zero-pad $v[n]$ to longer block length $N$. The spectrum is the same, we just have more samples.

Frequency Analysis with DFT

- Note that the ordering of the DFT samples is unusual.

$$V[k] = \sum_{n=0}^{N-1} v[n] W_N^{nk}$$

The DC sample of the DFT is $k = 0$

$$V[0] = \sum_{n=0}^{N-1} v[n] W_N^{nk} = \sum_{n=0}^{N-1} v[n]$$

- The positive frequencies are the first $N/2$ samples
- The first $N/2$ negative frequencies are circularly shifted

$$((-k))_N = N - k$$

so they are the last $N/2$ samples. (Use fftshift to reorder)
A 40 yo pt with a history of lower limb weakness referred for MRI screening of brain and whole spine for cord.

MRI sagittal T2 screening of dorsal region shows a faint uniform linear high signal at the center of the cord. The signal abnormality likely to represent:

2. Syrinx (spinal cord disease).
3. Artifact.

Answer: It's an artifact, known as truncation or Gibbs artifact.

Potential Problems and Solutions

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible Solutions</th>
</tr>
</thead>
</table>
| 1. Spectral error from aliasing Ch.4 | a. Filter signal to reduce frequency content above \( \Omega_s/2 = \pi/T \).
|                               | b. Increase sampling frequency \( \Omega_s = 2\pi/T \). |
| 2. Insufficient frequency resolution. | a. Increase \( L \). |
|                               | b. Use window having narrow main lobe. |
|                               | b. Increase \( L \). |
| 4. Missing features due to spectral sampling. | a. Increase \( L \). |
|                               | b. Increase \( N \) by zero-padding \( v[n] \) to length \( N > L \). |

Frequency Analysis with DFT

- Length of window determines spectral resolution
- Type of window determines side-lobe amplitude. (Some windows have better tradeoff between resolution-sidelobe)
- Zero-padding approximates the DTFT better. Does not introduce new information!