Announcements

• Last time:
  – FFT

• Today:
  – Frequency analysis with DFT
  – Windowing
  – Effect of zero-padding
The DFT can be used to analyze the spectrum of a signal.

It would seem that this should be simple, take a block of the signal and compute the spectrum with the DFT.

However, there are many important issues and tradeoffs:

- Signal duration vs spectral resolution
- Signal sampling rate vs spectral range
- Spectral sampling rate
- Spectral artifacts
Consider these steps of processing continuous-time signals:

\[ s_c(t) \xrightarrow{\text{Antialiasing lowpass filter}} x_c(t) \xrightarrow{H_{aa}(j\Omega)} x[n] \xrightarrow{\times} v[n] \xrightarrow{\text{DFT}} V[k] \]

\[ w[n] \]
Spectral Analysis with the DFT

Two important tools:
- Applying a window to the input signal – reduces spectral artifacts
- Padding input signal with zeros – increases the spectral sampling

Key Parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling interval</td>
<td>$T$</td>
<td>s</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>$\Omega_s = \frac{2\pi}{T}$</td>
<td>rad/s</td>
</tr>
<tr>
<td>Window length</td>
<td>$L$</td>
<td>unitless</td>
</tr>
<tr>
<td>Window duration</td>
<td>$L \cdot T$</td>
<td>s</td>
</tr>
<tr>
<td>DFT length</td>
<td>$N \geq L$</td>
<td>unitless</td>
</tr>
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<td>$N \cdot T$</td>
<td>s</td>
</tr>
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<tr>
<td>Spectral sampling interval</td>
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</tbody>
</table>
Filtered Continuous-Time Signal

We consider an example:

\[ x_c(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \]

\[ X_c(j\Omega) = A_1 \pi [\delta(\Omega - \omega_1) + \delta(\Omega + \omega_1)] + A_2 \pi [\delta(\Omega - \omega_2) + \delta(\Omega + \omega_2)] \]
Sampled Signal

If we sampled the signal over an infinite time duration, we would have:

\[ x[n] = x_c(t) \big|_{t=nT}, \quad -\infty < n < \infty \]

described by the discrete-time Fourier transform:

\[ X(e^{j\Omega T}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_c \left( j \left( \Omega - r \frac{2\pi}{T} \right) \right), \quad -\infty < \Omega < \infty \]

Recall \( X(e^{j\omega}) = X(e^{j\Omega T}) \), where \( \omega = \Omega T \) ... more in ch 4.
In the examples shown here, the sampling rate is $\Omega_s/2\pi = 1/T = 20$ Hz, sufficiently high that aliasing does not occur.
Block of $L$ Signal Samples
In any real system, we sample only over a finite block of $L$ samples:

$$x[n] = x_c(t)|_{t=nT}, \quad 0 \leq n \leq L - 1$$

This simply corresponds to a rectangular window of duration $L$.

Recall: in Homework 1 we explored the effect of rectangular and triangular windowing
Windowed Block of $L$ Signal Samples
We take the block of signal samples and multiply by a window of duration $L$, obtaining:

$$v[n] = x[n] \cdot w[n], \quad 0 \leq n \leq L - 1$$

Suppose the window $w[n]$ has DTFT $W(e^{j\omega})$.

Then the windowed block of signal samples has a DTFT given by the periodic convolution between $X(e^{j\omega})$ and $W(e^{j\omega})$:

$$V(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$
Convolution with $W(e^{j\omega})$ has two effects in the spectrum:

1. It limits the spectral resolution. – Main lobes of the DTFT of the window
2. The window can produce *spectral leakage*. – Side lobes of the DTFT of the window

* These two are always a tradeoff - time-frequency uncertainty principle
## Windows (as defined in MATLAB)

<table>
<thead>
<tr>
<th>Name(s)</th>
<th>Definition</th>
<th>MATLAB Command</th>
<th>Graph ($M = 8$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular Boxcar</td>
<td>$w[n] = \begin{cases} 1 &amp;</td>
<td>n</td>
<td>\leq M/2 \ 0 &amp;</td>
</tr>
<tr>
<td>Fourier</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Triangular</td>
<td>$w[n] = \begin{cases} 1 - \frac{</td>
<td>n</td>
<td>}{M/2 + 1} &amp;</td>
</tr>
<tr>
<td>Bartlett</td>
<td>$w[n] = \begin{cases} 1 - \frac{</td>
<td>n</td>
<td>}{M/2} &amp;</td>
</tr>
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<tr>
<td>Hann</td>
<td>( w[n] = \begin{cases} \frac{1}{2} \left( 1 + \cos \left( \frac{\pi n}{M/2} \right) \right) &amp;</td>
<td>n</td>
<td>\leq M/2 \ 0 &amp;</td>
</tr>
<tr>
<td>Hanning</td>
<td>( w[n] = \begin{cases} \frac{1}{2} \left( 1 + \cos \left( \frac{\pi n}{M/2 + 1} \right) \right) &amp;</td>
<td>n</td>
<td>\leq M/2 \ 0 &amp;</td>
</tr>
<tr>
<td>Hamming</td>
<td>( w[n] = \begin{cases} 0.54 + 0.46 \cos \left( \frac{\pi n}{M/2} \right) &amp;</td>
<td>n</td>
<td>\leq M/2 \ 0 &amp;</td>
</tr>
</tbody>
</table>
Windows

- All of the window functions $w[n]$ are real and even.

- All of the discrete-time Fourier transforms

$$W(e^{j\omega}) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} w[n]e^{-jn\omega}$$

are real, even, and periodic in $\omega$ with period $2\pi$.

- In the following plots, we have normalized the windows to unit d.c. gain:

$$W(e^{j0}) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} w[n] = 1$$

This makes it easier to compare windows.
Window Example

\begin{figure}
\centering
\includegraphics[width=\textwidth]{window_example}
\caption{Window Example}
\end{figure}

\textit{Miki Lustig UCB. Based on Course Notes by J.M Kahn Spring 2014, EE123 Digital Signal Processing}
Windows Properties

These are characteristic of the window type

<table>
<thead>
<tr>
<th>Window</th>
<th>Main-lobe</th>
<th>Sidelobe $\delta_s$</th>
<th>Sidelobe $-20 \log_{10} \delta_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rect</td>
<td>$\frac{4\pi}{M+1}$</td>
<td>0.09</td>
<td>21</td>
</tr>
<tr>
<td>Bartlett</td>
<td>$\frac{8\pi}{M+1}$</td>
<td>0.05</td>
<td>26</td>
</tr>
<tr>
<td>Hann</td>
<td>$\frac{8\pi}{M+1}$</td>
<td>0.0063</td>
<td>44</td>
</tr>
<tr>
<td>Hamming</td>
<td>$\frac{8\pi}{M+1}$</td>
<td>0.0022</td>
<td>53</td>
</tr>
<tr>
<td>Blackman</td>
<td>$\frac{12\pi}{M+1}$</td>
<td>0.0002</td>
<td>74</td>
</tr>
</tbody>
</table>

Most of these (Bartlett, Hann, Hamming) have a transition width that is twice that of the rect window.

**Warning:** Always check what’s the definition of $M$

Adapted from *A Course In Digital Signal Processing* by Boaz Porat, Wiley, 1997
Windows Examples

Here we consider several examples. As before, the sampling rate is \( \Omega_s/2\pi = 1/T = 20 \text{ Hz.} \)

**Rectangular Window, \( L = 32 \)**
Windows Examples

Triangular Window, \( L = 32 \)

- Triangular Window, \( L = 32 \)
- DTFT of Triangular Window
- Sampled, Windowed Signal, Triangular Window, \( L = 32 \)
- DTFT of Sampled, Windowed Signal

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Windows Examples

Hamming Window, $L = 32$

- **Hamming Window, $L = 32$**
- **DTFT of Hamming Window**
- **Sampled, Windowed Signal, Hamming Window, $L = 32$**
- **DTFT of Sampled, Windowed Signal**

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Windows Examples

Hamming Window, $L = 64$

Sampled, Windowed Signal, Hamming Window, $L = 64$

DTFT of Sampled, Windowed Signal

Hamming Window, $L = 64$

DTFT of Hamming Window

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Optimal Window: Kaiser

- Minimum main-lobe width for a given side-lobe energy %

\[
\frac{\int_{\text{sidelobes}} |H(e^{j\omega})|^2 d\omega}{\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega}
\]

- Window is parametrized with L and \( \beta \)  
  - \( \beta \) determines side-lobe level
  - L determines main-lobe width
Example

\[ y = \sin(2\pi 0.1992n) + 0.005 \sin(2\pi 0.25n) \quad | \quad 0 \leq n < 128 \]
Example

Hamming

Hann

Kaiser Beta=6

Kaiser Beta=9
Zero-Padding

- In preparation for taking an $N$-point DFT, we may zero-pad the windowed block of signal samples to a block length $N \geq L$:

\[
\begin{cases}
    v[n] & 0 \leq n \leq L - 1 \\
    0 & L \leq n \leq N - 1
\end{cases}
\]

- This zero-padding has no effect on the DTFT of $v[n]$, since the DTFT is computed by summing over $-\infty < n < \infty$.

Effect of Zero Padding

- We take the $N$-point DFT of the zero-padded $v[n]$, to obtain the block of $N$ spectral samples:

\[
V[k], \quad 0 \leq k \leq N - 1
\]
Consider the DTFT of the zero-padded $v[n]$. Since the zero-padded $v[n]$ is of length $N$, its DTFT can be written:

$$V(e^{j\omega}) = \sum_{n=0}^{N-1} v[n] e^{-j n \omega}, \quad -\infty < \omega < \infty$$

The $N$-point DFT of $v[n]$ is given by:

$$V[k] = \sum_{n=0}^{N-1} v[n] W_N^{kn} = \sum_{n=0}^{N-1} v[n] e^{-j(2\pi/N)nk}, \quad 0 \leq k \leq N - 1$$

We see that $V[k]$ corresponds to the samples of $V(e^{j\omega})$:

$$V[k] = V(e^{j\omega})|_{\omega=k\frac{2\pi}{N}}, \quad 0 \leq k \leq N - 1$$

To obtain samples at more closely spaced frequencies, we zero-pad $v[n]$ to longer block length $N$. The spectrum is the same, we just have more samples.
Frequency Analysis with DFT

- Note that the ordering of the DFT samples is unusual.

\[ V[k] = \sum_{n=0}^{N-1} v[n] W_N^{nk} \]

The DC sample of the DFT is \( k = 0 \)

\[ V[0] = \sum_{n=0}^{N-1} v[n] W_N^{0n} = \sum_{n=0}^{N-1} v[n] \]

- The positive frequencies are the first \( N/2 \) samples
- The first \( N/2 \) negative frequencies are circularly shifted

\[ ((-k))_N = N - k \]

so they are the last \( N/2 \) samples. (Use \texttt{fftshift} to reorder)
Frequency Analysis with DFT Examples:

Hamming Window, $L = 32$, $N = 32$

Sampled, Windowed Signal, Hamming Window, $L = 32$, Zero-Padded to $N = 32$

$N$-Point DFT of Sampled, Windowed, Zero-Padded Signal

Spectrum of Sampled, Windowed, Zero-Padded Signal

$|V[k]|$, $\omega_k = k\frac{2\pi}{NT}$

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Frequency Analysis with DFT Examples:

Hamming Window, $L = 32$, Zero-Padded to $N = 64$

Sampled, Windowed Signal, Hamming Window, $L = 32$, Zero-Padded to $N = 64$

N-Point DFT of Sampled, Windowed, Zero-Padded Signal

Spectrum of Sampled, Windowed, Zero-Padded Signal

$|V(\omega)|$ vs $\omega$ (Hz)

$|V[k]|$, $\omega_k = k2\pi/NT$
A 40 yo pt with a history of lower limb weakness referred for MRI screening of brain and whole spine for cord. MRI sagittal T2 screening of dorsal region shows a faint uniform linear high signal at the center of the cord. The signal abnormality likely to represent:

(1) Cord demyelination.
(2) Syrinx (spinal cord disease).
(3) Artifact.

**Answer**: Its an artifact, known as truncation or Gibbs artifact
- Length of window determines spectral resolution.

- Type of window determines side-lobe amplitude. *(Some windows have better tradeoff between resolution-sidelobe)*

- Zero-padding approximates the DTFT better. Does not introduce new information!
<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Spectral error from aliasing Ch.4</td>
<td>a. Filter signal to reduce frequency content above $\Omega_s/2 = \pi/T$. b. Increase sampling frequency $\Omega_s = 2\pi/T$.</td>
</tr>
<tr>
<td>2. Insufficient frequency resolution.</td>
<td>a. Increase $L$                                                                                                  b. Use window having narrow main lobe.</td>
</tr>
<tr>
<td>3. Spectral error from leakage</td>
<td>a. Use window having low side lobes. b. Increase $L$</td>
</tr>
</tbody>
</table>